The Fractional Fourier Transform as an Optical Transformation using Discrete Optics

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We develop an experiment to replicate the theoretical fractional Fourier transformation (FRT) as an optical Fourier transformation (OFT), using a spatially filtered HeNe laser, two plano-convex lenses and CCD camera. The fractional Fourier transformation is successfully replicated and we find excellent agreement with theoretical predictions.

1.0 Introduction

The Fourier transforming properties of a converging lens have long been known. The Fractional Fourier Transform (FRT) is recent development that is a continuous generalisation of the Fourier Transformation (FT) (Mendlovic et Al, 1993). The FRT was originally developed to model propagation of light the quadratically graded index media (notably optical fibres). It can however be shown to be present in even the simplest of optical systems, as we will show later in this article.

2.0 The Fractional Fourier Transform

Let our Fourier Transform operator be denoted by $\mathcal{F}$. We extend our notation by the use of an order $a$, as $\mathcal{F}^a$. $\mathcal{F}^1$ denotes the normal FT. $\mathcal{F}^2$ denotes applying the transform twice. It is clear to see that $\mathcal{F}^0$ is the identity operator which returns the original operand.

The linear FRT is most easily understood by staring with the discrete definition of the linear FT but with the additional inclusion of a factor $b$,

$$F_n = \sum_{k=0}^{N-1} f_k e^{\frac{2i\pi nk}{N}}.$$  \hfill (1)

However, the inverses of this formula may not be consistent unless $b$ is an integer relatively prime to $N$, such that $(b,N)=1$.

A more robust definition is as follows. The normal FT can be considered to be a rotation of $\pi/2$ in the space-frequency plane. The FRT can then be considered to be a rotation through any arbitrary angle forwards or backwards. A trivial proof can show that $\mathcal{F}^a = \mathcal{F}^0 = \mathcal{F}^{-a}$.

Without further explanation, we quote the most important formal definition of the FRT (Ozaktas, Zalevsky et al. 2001).

The $a$th order fractional Fourier transform is a linear operation defined by the integral

$$f_a(u) = \int K_a(u,u')f(u')du',$$

$$K_a(u,u') = A_a \exp\left[i\pi(\cot u'^2 - 2\csc u' \cot u + \cot u'^2)\right],$$

$$A_a = \sqrt{1 - i \cot \alpha},$$

$$\alpha = \frac{a\pi}{2}. $$  \hfill (2)

The reader is referred to (Ozaktas, Zalevsky et al. 2001) for a more rigorous treatment of the FRT.
2.1 Fourier Properties of a Lens

Consider the optical transforming configuration

\[ F_{\text{IG}} \]

![Diagram of an optical transforming configuration](image)

FIG 1 Object placed in front of lens.

The object, located a distance \( d_0 \) in front of the lens of focal length \( f \) is illuminated by a normally incident plane wave of amplitude \( A \) and distribution \( t_0 \). The Fourier spectrum of the light transmitted by the object is represented by \( F_0(\xi_x, \xi_y) \) and that of the light incident on the lens by \( F_i(\xi_x, \xi_y) \). The final amplitude distribution behind the lens is represented by \( U_f(x, y) \).

We consider here the Fraunhofer approximations\(^1\).

The light transmitted by the object and the light incident on the lens can be related by the Fresnel formula,

\[
F_i(\xi_x, \xi_y) = F_0(\xi_x, \xi_y)e^{-i\pi d_0(\xi_x^2 + \xi_y^2) / f}. \tag{3}
\]

The resultant light from the lens can be related to the incident light by means of the formula (Goodman 1968)

\[
U_f(x, y) = \frac{1}{i\lambda f} e^{i2f(x^2+y^2)} \int_{\Delta x, \Delta y} F_0(x_0, y_0) e^{-i\pi d_0(x_0^2+y_0^2) / f} d(x_0, y_0) \tag{4}
\]

By substituting equation (3) into equation (4), we have

\[
U_f(x, y) = \frac{1}{i\lambda f} e^{i2f(x^2+y^2) / f} F_0(x, y) \tag{5}
\]

\(1\) The Fraunhofer approximation states that the hypotenuse subtending points at one reference plane to that on another some distance away is parallel to the optic axis, its length therefore equivalent to the planar separation, and need not be derived by use of Pythagoras’s theorem.

Examination of (6) reveals that this is an exact Fourier transform relation (Goodman 1968).

The process of the optical transformation is most easily understood by the following illustration.

When illuminated, every point on the object (input image) emits a cone of rays towards the lens, which are brought to the parallel. Conversely, every parallel ray from the input image is converged to a point. The superposition of these two phenomena is mathematically equivalent to the Fourier transformation (c.f. Fig 2).

Additional considerations of this derivation are discussed in (Goodman 1968) as is the case of lens of finite aperture size which limits the maximal spatial frequency the lens is capable of remitting (vignetting).

2.2 Fractional Fourier Properties of a Lens

It can be shown that FT relation (6) generalises to the FRT (2), which gives the transformation for any arbitrary distance \( z \).\(^2\)

The relationship of distance \( z \) to order \( a \) is given by the relation (Lohmann 1995)

\[
z = f \sin \left( \frac{a\pi}{2} \right) \tan \left( \frac{a\pi}{4} \right) \tag{7}
\]

There is no analytical solution to (7), however we can still plot this function (MapleSoft 2004).

\(2\) This derivation is beyond the scope of this article. The reader is referred to [Lohmann, 1995 #3] for evidence, and [Ozaktas, 2001 #1] as supplement.
finding it to be distinctly nonlinear. The obeyed boundary conditions are \( z = 0, a = 0 \), and at \( z = f \) (\( f = 1.4m \) in this case), \( a = 1 \). The curve is approximately linear except in the region \( z < 0.4m \), where the curve becomes quadratic. These behaviours reflect well the continuous transition between Fresnel and Fraunhoffer diffraction zones.

We will proceed to show how this can be performed using bulk optics.

### 3.0 A FRT Discrete Optics Systems

We base our experiment on the original experiment that demonstrates the optical Fourier transform and extend it to test the FRT.

The experiment consists of a spatially filtered HeNe laser passed through two planoconvex lenses arranged symmetrically. The spatial filter spreads and eliminates all off axis and scattered light from the beam. The beam rays pass through the object in parallel.

The data is captured using a digital CCD.

The position of the CCD is varied relative to the apparatus.
3.1 The Object

We will use a cross hair as our object/input image. This was chosen as its transform resolved well on the apparatus when compared against other simple objects.

![Crosshair Image Transformed](image)

3.2 Experimental Method

The position of the CCD is varied in fixed intervals in the range $0 < z < f$ and images are captured using the CCD. This produces a sequence of images. These images are to be compared subjectively against computer generated images of the FRT to determine the order $a$.

3.3 Computer Generated Images

Images of the FRT of a crosshair were generated using MATLAB (MathWorks 2002). The algorithm is based upon the original described in (Ozaktas, Arikan et al. 1996) that computes the discrete FRT for a vector, however we have generalized it here for use with an $N$-dimensional system.

A sequence of images was generated in the range of order $0 < a < 1$ with interval $a_{n+1} - a_n = 1/300$.

The image order $a$ must then calibrated to the apparatus by means of a factor $f/z$. This ensures that the size of the crosshair that we have transformed during calculation is normalised to the size being transformed at distance $z$ from the lens.

4.0 Experimental Results

Comparing the computer generated images of known order against those captured by apparatus, we have judged subjectively the unnormalised order of the optical FRT at each distance $z$. Where it was difficult to accurately determine the order of a particular image, that image was not included in the data.

The images are shown below side by side for easy comparison. The left hand images are optical; those on the right are MATLAB.

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3 This operation is permitted as it commutes with the FRT. This is most easily seen in Wigner space, where the commutators are $f/z$ and $\begin{bmatrix} \cos(a\pi/2) & \sin(a\pi/2) \\ \sin(a\pi/2) & -\cos(a\pi/2) \end{bmatrix}$. 

$\begin{bmatrix} f/z & 0 \\ 0 & f/z \end{bmatrix}$
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4.1 Analysis of Data

We have plotted the normalised fractional order \( a \) against lens-CCD distance \( z \). The error in \( a \) represents one standard deviation in each direction \((a \pm \sigma)\).

We will attempt to investigate the linear region \( z > 0.4 \) (c.f. §2.2), so we eliminate any points outside this region and perform a least squares linear fitting operation.

This renders a relationship

\[
0.58 \pm 0.02 \cdot z + (0.23 \pm 0.05) = a
\]

with a \( \chi^2 \) value of 3.6, implying a good estimation of the errors.

The \( a \) intercept of \( 0.23 \pm 0.05 \) (not shown) and gradient of \( 0.58 \pm 0.02 \) strongly agrees with the
estimated intercept $a = 0.3$ and $\frac{da}{dz} = 0.5$, as evident from Fig 3.

5.0 Conclusion

Our graph agrees well for the modelling of the linear region of the curve of the shown in Fig 3.

We have shown that consecutive planes between a convex lens and its Fourier plane are fractional Fourier planes, in accordance with formal theories of Ozaktas (Ozaktas, Zalevsky et al. 2001).

Furthermore, we have shown that $z(f,a)$ distribution of Lohmann (Lohmann 1995) is well matched in the linear region $z \geq \frac{f}{3}$ for the behaviour of a single convex surface.

REFERENCES

MathWorks (2002). MATLAB.