Acoustic properties of periodically and quasi-periodically modulated waveguides

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Abstract

The occurrence of transmission band gaps in waveguides whose area is modulated in a locally periodic and quasi-periodic fashion is demonstrated, both theoretically and experimentally. A band structure approach is utilised, and band gaps are shown to form at the edge and centre of the Brillouin zone dependent on the exact configuration of the structure. The features of the band gap become more defined as the number of periods (or layers in a quasi-periodic system) is increased and the band gaps are shown (for above approximately 8 periods) to have very steep slopes and flat reject bands. Central defects are introduced into the periodic arrangement giving rise to perfect transmission defect peaks within the band gap due to spatial localization at the defect. These peaks have high Q-factors, which increase with the number of periods and proximity of the peak to the centre of the band gap due to the higher degree of spatial confinement in these cases. The length of the defect controls the location of the peak, and both donor-like (longer than the original section) and acceptor-like (shorter than the original section) defects are considered. The defect peak is shown to occur at the centre of the band gap when the defect is exactly double (donor-like) or half (acceptor-like) the length of the equivalent non-defect section. Structures based on Fibonacci sequences are shown to develop wide pseudo-band gaps interrupted by transmission (defect-type) peaks around the centre of the band gap. Other quasi-periodic structures with lower degrees of periodicity show less defined pseudo-gaps. The form of the defect peaks and the scaling nature of the transmission from the Fibonacci based structures are demonstrated experimentally and good agreement is seen with the theoretical predictions. Heterostructures are also investigated, and it is shown that the band gap can be increased by combining two or more structures with complementary band gaps. Fractal-type structures are seen to introduce extra gaps in transmission that make these effective for use in heterostructures. A genetic algorithm is used to numerically minimise the transmission from a modulated structure by optimising the structure layer thicknesses. Short structures result, although small peaks in transmission in the frequency range of interest are also seen. These structures, coupled with the results of the quasi-periodic structures, show that periodicities of between two and three layer thicknesses are important for the creation of band gaps.
## Abbreviations and Common Symbols

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>ABG</td>
<td>Acoustic band gap</td>
</tr>
<tr>
<td>ACF</td>
<td>Auto-correlation function</td>
</tr>
<tr>
<td>BZ</td>
<td>Brillouin zone</td>
</tr>
<tr>
<td>FS</td>
<td>Fibonacci sequence</td>
</tr>
<tr>
<td>FT</td>
<td>Fourier transform</td>
</tr>
<tr>
<td>MLS</td>
<td>Maximum length sequence</td>
</tr>
<tr>
<td>PBG</td>
<td>Photonic band gap</td>
</tr>
<tr>
<td>TM</td>
<td>Transfer matrix</td>
</tr>
<tr>
<td>TMM</td>
<td>Transfer matrix method</td>
</tr>
<tr>
<td>$\alpha_r$</td>
<td>Intensity reflection coefficient</td>
</tr>
<tr>
<td>$\alpha_t$</td>
<td>Intensity transmission coefficient</td>
</tr>
<tr>
<td>$\rho_0$</td>
<td>Density of air (assumed 1.21 Kgm$^{-3}$)</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Ratio of the areas of the standard and perturbed waveguide $S_1/S_2$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Bloch phase</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Cosine of the Bloch phase</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Extinction ratio of a peak</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Angular frequency</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>Periodic repeat distance</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Acoustic pressure</td>
</tr>
<tr>
<td>$2a$</td>
<td>Waveguide perturbation length</td>
</tr>
<tr>
<td>$c$</td>
<td>Speed of sound in air (assumed 343 ms$^{-1}$)</td>
</tr>
<tr>
<td>$f$</td>
<td>Frequency</td>
</tr>
<tr>
<td>$f_c$</td>
<td>Centre frequency of band gap (design frequency)</td>
</tr>
<tr>
<td>$k$</td>
<td>Wavenumber</td>
</tr>
<tr>
<td>$\ell$</td>
<td>Spacing of waveguide perturbations</td>
</tr>
<tr>
<td>$Q$</td>
<td>Quality factor (Q-factor) of a peak</td>
</tr>
<tr>
<td>$S$</td>
<td>Cross-sectional area</td>
</tr>
<tr>
<td>$u$</td>
<td>Particle velocity</td>
</tr>
<tr>
<td>$U$</td>
<td>Volume velocity</td>
</tr>
<tr>
<td>$P$</td>
<td>Transfer matrix relating adjacent unit cells</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction and literature review

1.1 Introduction

Structures which have a regular distribution of scattering centers have been seen to possess a distinct and interesting array of acoustical properties, perhaps most strikingly regions of frequency in which sound cannot propagate through the structure. Such phononic (or sonic) crystals have been the subject of intense investigation in recent years, both in their own right as acoustical systems but also as an acoustical analogue to the electronic behaviour of crystal solids and the optical behaviour of so-called photonic crystals. As such, this topic forms an interesting bridge between the often uncorrelated disciplines of solid state physics, optics and acoustics.

In fact, as will be discussed, it is concepts from solid state physics which form the foundations of much of the theory of these structures. Therefore many of the features of phononic crystals are discussed most naturally in nomenclature that is common in solid state physics, but is not routinely used in acoustics. An attempt to elucidate the various terms used will be made throughout this work. However, for the sake of brevity, a detailed explanation of the terms will not be included. For a fuller introduction to these terms and their context, the reader is referred to the introductory solid state physics texts of Ashcroft and Mermin [1] and Kittel [2].

The following sections in this chapter present a detailed literature review summarising the major theories and developments in the field of periodic and quasi-periodic media, and the objectives of and motivations behind the work presented in this thesis.
1.2 Area modulation of waveguides

A rectangular waveguide offers (under certain assumptions) a suitable system in which to model the propagation of plane waves through a 1-dimensional lossless medium. As such, they are systems which allow analytically tractable theoretical models to be verified with an experimental system which is fairly easy to construct.

A discontinuity in the cross-sectional area of a rigid waveguide produces an impedance discontinuity causing incident plane waves to be partially reflected at the interface of the two sections of waveguide. It has long been known (see, for example, the classic texts by Beranek [3] and Kinsler et al. [4]) that structures incorporating such discontinuities can exhibit acoustical filtering effects. More advanced filters can be constructed using the principles of ladder-type networks familiar from electrical filters, and in this form, the filtering properties of periodically modulated waveguides are well documented [4, pp. 242–244]. Indeed, Kinsler et al. [4] discuss how, for example, a periodic structure can be used to increase the sharpness of the cut-off of the filter with respect to that of a single element. They suggest such periodic structures can be incorporated into many mechanoacoustic systems such as forced air heating, exhaust systems and mufflers.

These systems are often modelled assuming a lumped, rather than distributed, behaviour, with the resulting analysis only valid at low frequencies. This is true of the analysis of Kinsler et al. [4] limiting the validity of their findings. This lumped-parameter approach obscures many of the properties of such systems, which have become of considerable interest over recent years. Alternative methods of analysis, such as using a transfer matrix method (TMM), are therefore desirable, and these will be discussed in further detail subsequently.

1.3 Energy bands and photonic crystals

A periodically modulated waveguide is actually part of a larger group of systems known as phononic (or sonic) crystals, which are systems which exhibit a so-called acoustic band gap (ABG). These are the acoustical equivalent of the solid state crystal and the optical photonic crystal.

In order to consider the phononic crystal in detail, it is necessary to discuss the
solid state crystal. A crystal lattice is formally described by a set of delta functions periodically distributed throughout space (and technically infinite in extent). Note, the crystal lattice is a mathematical abstraction, with the crystal itself formed when a *basis* (a small set of atoms describing the *unit cell* of the crystal) is attached to each lattice point. The behaviour of an electron in a crystal can be described by treating the lattice as a periodic potential (provided by the atoms situated in the crystal). In their seminal work, Kronig and Penney [5] showed that this results in the formation of allowed and forbidden bands in energy, a result which is the basis of all modern solid state physics. This introduces the important concept of the *band gap* separating regions of allowed energy, which describes not only the existence of metals and insulators, but also the important class of semiconductors and semi-metals. The implications of electronic energy bands in crystal systems are treated by numerous introductory texts on solid state physics, and the interested reader is referred to, for example, those by Ashcroft and Mermin [1] and Kittel [2].

The formation of energy bands described above is due to interference of waves (in this case, the quantum mechanical electronic wave functions) resulting when they are scattered from the periodic potential of the lattice. This effect is not limited to the electronic properties of solids, but can occur for purely classical waves when in a medium with some scattering mechanism arranged in a periodic manner. Thus, in principle, acoustical, electromagnetic, mechanical or even water waves should display this phenomenon.

In general, the optical case (where the scattering mechanism is provided by changes in the refractive index of the medium) has advanced first, driven by numerous practical applications, with the acoustical case being investigated somewhat later. Indeed, the simplest form of a photonic crystal (a medium in which the refractive index is modulated periodically), a structure consisting of planes of high and low index material, can be traced back to Lord Rayleigh in 1887 [6, 7]. He showed that any such periodic media would contain band gaps where light could not be transmitted through the medium, although he did not use this terminology. Although his work touched on some aspects of band theory, he did not peruse these, instead opting for a more direct description based on reflections from each layer of the medium [8]. This structure is now commonly known as a Bragg stack or Bragg reflector.
The detailed study of these (and higher dimensional counterparts) as photonic crystals was, however, not substantially investigated until the potential use for such structures in the control of light was demonstrated theoretically in two (independent) papers of Yablonovitch [9] and John [10] in 1987. A periodic thin-film interference filter, of the type investigated by Lord Rayleigh [6], was treated as a one-dimensional photonic crystal by Chigrin and Sotomayor Torres [11] in 2000. They showed that, when analysed as a photonic crystal, the familiar wavelength regions of high reflection for the structure could be characterised as forbidden energy bands, in an analogous way to the electronic energy bands in solids discussed above. The allowed energy bands are separated by a full photonic band gap (PBG) through which light cannot propagate, leading to total reflection when the light is in the frequency range of the PBG. They showed that analysis in terms of photonic crystals allows greater flexibility in the design of structures utilising a periodic modulation of the refractive index for use in optoelectronic applications.

1.4 Phononic crystals

In 1991, the theoretical physicist Jonathan Dowling [12] considered the case of an infinitely long tube of fluid with periodic small increases in density. Solving the Helmholtz equation (the time independent acoustic wave equation) in one-dimension in the presence of a periodically varying density, and invoking the Bloch-Floquet theorem\(^ {2}\) Dowling was able to show that a complete acoustic band gap (ABG) should exist (in one-dimension) in analogy to the PBG of photonic crystals and the electronic band gap of solids. Dowling suggested that the existence of band gaps should not be limited to a one dimensional system. Although his analysis was not directly applicable to systems of higher dimension, he argued that the analogy between the scalar nature of the sound field and the scalar nature of the quantum mechanical electron wavefunctions that exhibit electronic band structure in regular three-dimensional lattices indicate that an ABG should be realizable in higher-dimensions also. This model did, however, require the density of the actual fluid medium to be varied periodically, and did not consider the possibilities of some other generation of scattering mechanism. Correspondingly the general interfer-

\(^{1}\)The acoustic analogy of this one-dimensional photonic crystal will be investigated in this work.

\(^{2}\)The Bloch-Floquet theorem governs the properties of waves and the solution of wave equations in the presence of a periodically varying medium. It will be discussed in further detail in Section 2.2.
ence nature of the transmission properties of the structure was not made apparent.

The theoretical solution was extended to two dimensions by Sigalas and Economou [13] in 1993. They assumed a system comprising infinite cylinders parallel to the $z$-direction of the structure, embedded in a periodic manner. These cylinders have a different density to the surrounding medium, and it is this periodic change in density that is responsible for the resulting band structure. They showed that such a band structure would result by solving the elastic wave equation incorporating a Bloch wave function solution (as for Dowling’s treatment of the 1-dimensional case). It was a structure of this type that provided one of the first experimental verifications of a phononic crystal. Martínez-Sala et al. [14] demonstrated that a sculpture by Eusebio Sempere, exhibited at the Juan March Foundation in Madrid, consisting of an array of periodically spaced metal rods, exhibited a distinct attenuation in transmission at 1670 Hz consistent with theoretical predictions for a band gap at this frequency, although the exact nature of the sculpture prevented a full gap being observed. Indeed, Kushwaha [15] showed that the filling fraction (the fraction of the sample consisting of the scattering rods) was too small to exhibit complete ABGs, instead exhibiting a pseudogap resulting from a reduction in the density of states. This therefore did not represent a rigorous test of the theoretical predictions of band gap formation, although it did inspire future experimental work in this area.

A similar (although this time custom engineered) structure consisting of parallel regularly spaced cylindrical rods was analysed by Robertson and Rudy [16]. They measured the properties of the structure by an impulse response method. An excitation signal that was derived from the second derivative of a Gaussian was found to give sufficiently broad spectral coverage, and fourier transform techniques were used to determine the transmission through the structure. However, they were also able to use the phase data of the fourier transform to display anomalous dispersion (where the refractive index of the sample is dependent on frequency) within the band gaps. From the phase data, they were also able to generate dispersion relations for acoustic waves propagating in the structure, allowing the band structure to be (partially) mapped out. This represents an important experimental technique in the analysis of phononic crystals. Despite these advances in experimental techniques, their transmission results were rather noisy, and the resulting band gaps were not well formed. Furthermore, no theoretical results were
shown to allow comparisons with the experimental data.

Sigalas [17] extended his two dimensional theory to incorporate defects (when the perfect periodicity of the structure is broken). The periodic theory has also been extended to three dimensions by Kushwaha et al. [18] and Kushwaha and Halevi [19] have shown that a cubic array in air of spherical balloons filled with hydrogen exhibit ABGs when the balloons are arranged in the body-centered cubic and face-centered cubic arrays (for a description of crystal structures, the reader is referred to, for example, Hammond [20]). These theoretical results, however, lacked any experimental verification.

The case of one-dimensional phononic crystals fabricated from acoustic waveguides (upon which this work is based) has received reasonable interest over the past decade. In addition to its potential use as a structure for, for example, acoustic filtering, the one-dimensional waveguide provides a novel system for investigating the effects of phononic crystals in a simplified one-dimensional case, which provides insight for the considerably more complex higher dimensional systems.

Bradley [21] treated the case of a periodic waveguide in some detail, where the waveguide was taken to be composed of localized scatterers connected by uniform waveguide sections, all of which are assumed to be rigid and isothermal. The previous work on one-dimensional phononic crystals of Dowling [12] assumed a periodic modulation in the density of the medium as discussed above. No such assumption was made in this case, with any scattering mechanism, such as a change in cross-sectional area, allowed. By applying equations of conservation of mass, momentum and entropy in the waveguide, and by application of boundary conditions and the fact that the boundary conditions are invariant under translations $r \rightarrow r + n\Lambda \hat{e}_z$ where $n$ is an integer, $\Lambda$ is the periodic repeat length of the waveguide and $\hat{e}_z$ is the unit vector along the waveguide, Bradley showed a Bloch-Floquet like theorem is applicable to the case under consideration. The existence of ABGs follow as a direct result. Thus, it can be seen that the inclusion of a change in density in the structure is not necessary for the creation of band gaps, with the important feature being the presence of some scattering mechanism. In a companion paper, Bradley [22] also showed the existence of these band gaps experimentally. He considered a rectangular air filled waveguide with periodic rigidly-terminated side branches and obtained good agreement with his theoretical
predictions of the parameters of the Bloch waves. However, the transmission through such a structure (arguably its most important parameter) was not considered, and so many details of the theory could not be investigated.

The theoretical descriptions discussed above all invoked the Bloch-Floquet theorem in their solution. This requires that the structure being considered is infinite in extent so that the periodicity is perfect. This assumption is generally valid for the electronic band structure calculations of a crystal as millions of atomic layers exist within a macroscopic sample. For a phononic crystal, the number of layers will be much smaller and so the system is only *locally periodic* and Bloch's theorem cannot be directly applied. Griffiths and Steinke [23] consider the case of locally periodic media, and show that, in one-dimension, locally periodic systems exhibit features of band structure (regions of high and low transmission) even for a relatively small number of periods. Furthermore, they show that the band structure develops quickly as the number of periods increases. In addition to deriving a closed form simple solution to the locally periodic case (based on a transfer matrix approach), they also show that the general solution is valid for a large number of cases, including quantum mechanics, mechanical waves (both acoustical waves in air and mechanical waves along a rod or string), water waves and electromagnetic waves. In each case, local boundary conditions and equations governing the type of wave in each medium can be used to derive coefficients for the general solution. This indicates that the band gaps resulting from a periodic (local or global) medium results from the interference of waves, the exact type of which is not important. However, again the results were entirely theoretical, and so experimental verification of the properties of such systems was not achieved.

If a simple expansion (or contraction) of the waveguide is used as the scattering element, the size and position of the band gap can be partly controlled by adjusting the length of the scattering sections. If the length of the scattering section is set equal to the length of the 'normal' waveguide sections, the resulting system is the exact acoustic analogue to the optical Bragg reflector discussed above. Such a system was investigated experimentally by Munday *et al.* [24] who showed that regions of zero transmission occurred at approximately the expected band gap frequencies. Theoretical predictions for the system were also shown. However, they were plotted in different graphs with different scales, thus preventing accurate comparison of the measured results with the
theoretical predictions. Furthermore, the experimental results were shown as transmitted amplitudes through the waveguide with and without the modulated structure present, whereas the theoretical results were presented as transmission coefficients (ratio of the previous quantities) again preventing accurate comparison of the results.

They extended the system to consider the effects of replacing the central section of the structure with an element of a different length. They showed that by incorporating this *defect* into the structure, a defect mode is opened up within the band gap, allowing a narrow region of perfect transmission surrounded by the zero transmission of the band gap region. Theoretically, these defect modes are very sharp, and have potential applications for use as very narrow-band filters. By changing the length of the defect, the position of the perfect transmission peak can be adjusted, giving some degree of control over the acoustic properties of the structure. However, whilst their experimental results did reveal the existence of defect modes, they were seen to be much wider and of much lower transmission than the theoretical predictions, and the results were somewhat corrupted by noise. This was judged to be due to inaccuracies in the construction of the waveguide system. More accurately constructed waveguide systems therefore have the potential to give results which more accurately match the theoretical predictions. Again, the theoretical and experimental results were shown in different forms, preventing accurate comparison of experiment and theory. Note, this system is analogous to the optical interference filters described by Brooker [25, pp. 140–141].

A variety of other phenomena are associated with photonic and phononic crystals. These include tunnelling through the forbidden region leading to a significant increase in the group velocity of the tunnelling wave, negative refraction and focusing of the wave. In the acoustical case, these properties have had some investigation for small scale phononic crystals operating in the ultrasound regime [26, 27] but have had little investigation on larger scale phononic crystals that operate in the audible range. The exception to this is the article by Robertson *et al.* [28] who show that a tunnelling pulse does indeed display an increase in group velocity up to approximately twice the speed of sound in air in a waveguide with periodic side branches.
1.5 Quasi-periodic structures

Anderson [29] first showed that electronic localization (spatial confinement of electrons in a crystal) in random media was a wave phenomena. This led to considerable interest in the localization of electromagnetic and acoustic waves in random media. This has been studied in the acoustic case by Baluni and Willemsen [30], Sornette and Legrand [31] and, more recently, Luan and Ye [32] all for one-dimensional media. These one-dimensional systems are preferred for the study of localization as the theory can be implemented using transfer matrix techniques, and is simple enough to be implemented with minimal approximations. The above named authors have shown that localization occurs in such systems, with pseudo-band gaps occurring. Thus the acoustical (and equivalently optical) case can be used for investigation of the fundamental properties of localization, having distinct advantages over the electronic investigation, where electron-electron and spin-orbit interactions must be incorporated (significantly complicating the investigation) and the system is not truly one-dimensional.

Localization has also been seen to occur in quasi-periodic lattices. These have long been studied in solid state physics [2] and can be seen as occurring somewhere between the ideal fully periodic case and complete disorder. A quasi-periodic system based on the Fibonacci number sequence (discussed in Section 6.1.1) was first investigated by Merlin et al. [33]. These structures have since received considerable interest in the optical domain for the study of localization in quasi-periodic (rather than disordered) media. Kohmoto et al. [34] first showed that this process occurs. They applied a simple electromagnetic theory to show that pseudo-band gaps occur in the transmission spectra of Fibonacci multilayers of differing refractive index. They showed that the transmission spectrum was multi-fractal and that the transmission has a scaling property (this will be discussed in further detail below), providing evidence of the localization of the waves. These features were broadly verified experimentally by Gellermann et al. [35] in 1994.

Following these advances, various practical applications have been considered incorporating these Fibonacci structures. For example. Peng et al. [36] have shown that if a Fibonacci structure is mirrored to create a symmetrical structure, perfect transmission peaks result within the pseudo-gaps. These are effectively defect modes (similar to those discussed above) and result from the symmetry of the system. Dong et al [37] have also shown that by combining Fibonacci structures and periodic structures into a
single heterostructure it is possible to increase the bandwidth of the zero-transmission region (i.e. increase the width of the band gap of the system).

The acoustical work in this area lags somewhat behind the optical work. Some work has been performed on the transmission of acoustic phonons (the quantum particle of vibrational energy in solids) in Fibonacci superlattices [38, 39] where discrete dips in transmission have been seen, but studies in the audible frequency range are rather lacking. Pseudo-band gaps have been shown (theoretically) to occur in a tube loop structure (that is similar to a modulated waveguide) by Aynaouet al. [40] in 2005 and the existence of defect modes within the pseudo-gap is also predicted. The scaling property seen in the optical case is also seen to occur for this system. However, these results did not include any experimental verification. There has not as yet, to the best of the authors knowledge, been any subsequent experimental (or theoretical) work on quasi-periodic modulation in an acoustic system.

1.6 Objectives and motivation

This work considers the acoustic properties of periodically and quasi-periodically modulated waveguides. As can be seen from the literature review presented above, the work in this field has been somewhat fragmented to date. This work aims to consolidate some of the existing effects discussed above, and extend some ideas forward in an acoustical context. In particular, the lack of accurate experimental data to compare in detail with theoretical predictions represents a significant omission in the studies to date (both for periodic and quasi-periodic systems) and this will be addressed in this work. Further, there is an absence of work on combining structures to improve their acoustical properties, and this will also be investigated.

Initially, this work will aim to characterise (theoretically) a simple locally periodic waveguide system, with periodic expansions or contractions in cross-sectional area of the waveguide. A Bragg stack type design will be considered where the length of the perturbations (along the axis of the waveguide) is equal to the length of the unperturbed sections. The effects of varying these lengths and of the change in cross-sectional area of the sections will be considered in the context of the acoustical transmission of the system and the associated band structure. The effects of changing the relative length of the perturbed sections with respect to the unperturbed sections will also be considered. The
theoretical effects will be compared with the results of an experimental investigation.

This work will then aim to characterise a defect state in this periodic array. Again, theoretical predictions and experimental results will be compared. Careful experimental design will be applied in an attempt to achieve results that are in closer agreement to the theoretical predictions than were found by Munday et al. [24].

Quasi-periodic systems will be investigated. Theoretical predictions will be considered, and an attempt will be made to provide experimental verification of these predictions. Finally, heterostructures will be considered with a view to possible practical applications of such structures.

The motivation of this work is two-fold. Firstly, the waveguide structures under investigation have potential application as acoustic filters and mirrors. An obvious possible use is in acoustic noise control – silencers in forced air heating, exhaust systems and mufflers, where an air flow path must be provided, but it is desired to reduce the acoustic noise transmitted. There may also be more subtle potential uses for these or similar one-dimensional periodic structures, for example in ultrasonics, where the required scales of the structure are much smaller, and may therefore be more easily realisable.

Secondly, and perhaps more importantly, these systems are of interest from a fundamental physics viewpoint. In a periodic arrangement, they allow the investigation of the formation of band gaps in locally periodic media, and the potential investigation of associated effects such as an increase in group velocity and focusing effects. In a quasi-periodic arrangement, they represent a model system with which to investigate some fundamental properties of the localization of waves. These features are of interest directly in a one-dimensional case, but also are of interest as lower dimensional analogues of the considerably more complex two- and three-dimensional cases, where the multiple scattering events are often not amenable to a simple theoretical treatment.

The structure of this thesis is as follows. This chapter has presented a detailed literature review of the subject, and has presented the objectives for and motivation behind this work. Chapter 2 presents an introduction to the general theory of transfer matrices and Bloch waves utilised in this work and Chapter 3 discusses the experimental arrangement implemented. Chapters 4 – 7 present the (theoretical and experimental) results of periodic systems, periodic systems incorporating defects, quasi-periodic sys-
tems and heterostructures investigated in this work, and the relevant discussions of these results. The systems are compared and contrasted in Chapter 8 and the conclusions and suggestions for further work are presented in Chapter 9.
Chapter 2

Theory and modelling

2.1 Transfer matrix theory

There are many methods with which to model the acoustic response of a one-dimensional waveguide. These include calculating its Green function utilising interface response theory [40], iterative approaches based on Fresnel’s equations [24], eigenvalue analysis [21] and transfer matrix methods (TMM) [23, 30, 38]. Of these, the TMM is particularly flexible, and will be considered further in this work.

2.1.1 Scattering from a single perturbation

The TMM formulation utilised here relates the total pressure and volume velocity at two points in the medium. This is convenient for use in this work, and also allows the easy inclusion of extra impedance contributions, in addition to the modulation of the waveguide, if required.

Consider an arbitrary scattering object in an infinite waveguide. The scattering object (for instance a simple expansion in the waveguide) is assumed to exist over an interval \((x_1, x_2)\). In the waveguide, on either side of the scattering object, a positive and negative travelling harmonic wave are assumed to exist, with an acoustic pressure described by the wavefunction\(^1\)

\[
\Psi(x, t) = \psi(x)e^{j\omega t}
\]  

\(^1\)The pressure is notated \(\Psi\) rather than \(p\) to stress the relation with the electron wavefunction (normally notated \(\Psi\)) in quantum mechanics.
where the spatial component

\[
\psi(x) = \begin{cases} 
\hat{A}e^{ikx} + \hat{B}e^{-ikx} & (x < x_1) \\
\hat{C}e^{ikx} + \hat{D}e^{-ikx} & (x > x_2)
\end{cases}
\] (2.2)

where \( k = \frac{\omega}{c} \) is the wavenumber (where \( \omega \) is the angular frequency and \( c \) the speed of sound in air) and \( \hat{A}, \hat{B}, \hat{C} \) and \( \hat{D} \) are (complex) amplitude coefficients. Assuming plane wave propagation, application of Euler’s equation gives the equivalent volume velocity relations

\[
U(x, t) = \begin{cases} 
\frac{S(x)}{\rho_0 c} \left( \hat{A}e^{ikx} - \hat{B}e^{-ikx} \right) e^{j\omega t} & (x < x_1) \\
\frac{S(x)}{\rho_0 c} \left( \hat{C}e^{ikx} - \hat{D}e^{-ikx} \right) e^{j\omega t} & (x > x_2)
\end{cases}
\] (2.3)

where \( S(x) \) is the cross-sectional area of the waveguide at \( x \) and \( \rho_0 \) is the density of air. The wavefunction in the region \((x_1, x_2)\) is at present undefined. It is convenient to relate the pressure and volume velocity at \( x = x_1 \) to those at \( x = x_2 \) by a transfer matrix (TM), \( M \)

\[
\begin{pmatrix} \Psi(x_1, t) \\ U(x_1, t) \end{pmatrix} = M \begin{pmatrix} \Psi(x_2, t) \\ U(x_2, t) \end{pmatrix}.
\] (2.4)

Consider the specific case of an abrupt expansion of the waveguide, represented schematically in Fig. (2.1). The expansion (of cross sectional area \( S_2 \)) is assumed to exist in the region \( (-a, a) \) so that \( x_1 = -a \) and \( x_2 = a \) in the above expressions. A change of cross-sectional area in the waveguide causes an impedance discontinuity which will lead to reflected waves. The spatial component of the acoustic pressure on either side of the expansion is given by Eqn. (2.2), and that in the region \( (-a, a) \) is assumed to be composed of right- and left-wards travelling plane waves with amplitudes \( \hat{E} \) and

![Figure 2.1: A schematic representation of the change in area at an abrupt expansion in a waveguide.](image-url)
\( F \) respectively giving
\[
\Psi(x) = \left( \hat{E}e^{ikx} + \hat{F}e^{-ikx} \right)e^{j\omega t} \quad (-a < x < a).
\] (2.5)

Note, any acoustic losses in the system are neglected, and the wavenumber is therefore given by the real quantity \( k = \frac{\xi}{c} \) as before.

Plane wave propagation in a rigid waveguide can be easily represented using this approach, and the analysis is presented in Appendix A. The resulting transfer matrix (TM) relating pressure and volume velocity at either end of the perturbation is given by
\[
\begin{pmatrix}
\Psi(-a) \\
U(-a)
\end{pmatrix} =
\begin{pmatrix}
\cos(2ka) & j\frac{\rho c}{S_2} \sin(2ka) \\
\frac{S_2}{\rho c} \sin(2ka) & \cos(2ka)
\end{pmatrix}
\begin{pmatrix}
\Psi(a) \\
U(a)
\end{pmatrix}
\] (2.6)

where \( S_2 \) is the cross-sectional area of the perturbed section of waveguide defined over the interval \((-a, a)\) as discussed. Clearly, this transfer matrix is unimodular (has determinant 1), which will be of importance subsequently. Note, Griffiths and Steinke [23] show that this waveguide expansion is entirely analogous to the quantum mechanical case of a particle governed by the Schrödinger equation encountering a rectangular potential. This analogy extends to the case of a periodic potential.

### 2.1.2 Multiple cells

It is necessary to extend the analysis to the case of multiple scattering elements. Consider a periodic system of the type shown schematically in Fig. (2.2). A unit cell of a crystal is the region of space that, when repeated with the periodicity of the lattice, fills space completely. In this case, the unit cell consists of a single period of the repeated structure as shown in Fig. (2.2). Consider the wavefunction (pressure) in the \( n^{th} \) unit

![Figure 2.2: Schematic representation of \( N \) periodic repetitions (of period length \( \Lambda \)) of abrupt expansions (of length \( 2a \)) and contractions in a waveguide.](image-url)
cell of the structure

\[ \Psi_n(x) = \left( \hat{A}_n e^{ik(x-n\Lambda)} + \hat{B}_n e^{-ik(x-n\Lambda)} \right) e^{j\omega t} \]  

(2.7)

where \( \Lambda \) is the period of the structure, \( \hat{A}_n \) and \( \hat{B}_n \) are the amplitude of the right and left travelling wave respectively and the phase acquired due to propagation between the unit cells has been included explicitly. The pressure and volume velocity at the right of each unit cell can be related by incorporating an extra transfer matrix of the form in Eqn. (2.6) giving

\[
\begin{pmatrix} \Psi_n \\ U_n \end{pmatrix} = \begin{pmatrix} \cos(2ka) & j\frac{\rho c}{\rho_0 c} \sin(2ka) \\ j\frac{S_2}{\rho_0 c} \sin(2ka) & \cos(2ka) \end{pmatrix} \begin{pmatrix} \cos(k\ell) & j\frac{\rho c}{\rho_0 c} \sin(k\ell) \\ j\frac{S_1}{\rho_0 c} \sin(k\ell) & \cos(k\ell) \end{pmatrix} \begin{pmatrix} \Psi_{n+1} \\ U_{n+1} \end{pmatrix}
\]

(2.8)

where

\[ P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \]

(2.9)

with

\[
P_{11} = \cos(2ka) \cos(k\ell) - \frac{S_1}{S_2} \sin(2ka) \sin(k\ell)
\]

(2.10)

\[
P_{12} = j\rho_0 c \left( \frac{1}{S_1} \cos(2ka) \sin(k\ell) + \frac{1}{S_2} \sin(2ka) \cos(k\ell) \right)
\]

(2.11)

\[
P_{21} = j\rho_0 c \left( S_2 \sin(2ka) \cos(k\ell) + S_1 \cos(2ka) \sin(k\ell) \right)
\]

(2.12)

\[
P_{22} = \cos(2ka) \cos(k\ell) - \frac{S_2}{S_1} \sin(2ka) \sin(k\ell)
\]

(2.13)

and

\[ \ell = (\Lambda - 2a) \]

(2.14)

is the length of each ‘normal’ waveguide section between each perturbation. Thus, \( P \) is the transfer matrix that relates the pressure and volume velocity in the \( n \)th cell to the \((n+1)\)th cell. For the waveguide represented in Fig. (2.2), the transfer matrices relating all unit cells are equal (due to the periodicity of the structure). However, this is not a necessary constraint for the method. Letting the transfer matrix relating the \( n \)th and
\( (n + 1) \text{th} \) cell be \( P_n \), the pressure and volume velocity at the left of the structure can be related to those at the right of the structure by the simple concatenation of the TMs

\[
\begin{pmatrix} \Psi_0 \\ U_0 \end{pmatrix} = \prod_{i=0}^{N-1} P_i \begin{pmatrix} \Psi_N \\ U_N \end{pmatrix}.
\]  

(2.15)

In the current case, however, all the transfer matrices are equal as the structure is (locally) periodic, and so

\[
\begin{pmatrix} \Psi_0 \\ U_0 \end{pmatrix} = P^N \begin{pmatrix} \Psi_N \\ U_N \end{pmatrix}
\]  

(2.16)

where the transfer matrix is simply raised to the power of the number of periods in the structure. It would be possible, therefore, to calculate the relation between the amplitude coefficients by multiplying this matrix by itself \( N \) times, but for large \( N \) this is rather inefficient leading to long computation modelling times. Instead, by noting that the transfer matrix \( P \) is unimodular (has determinant 1, see Eqn. (2.9)) the Cayley-Hamilton theorem can be used to determine a closed form expression for the \( N \)th power of the matrix in terms of Chebychev polynomials, as discussed in Appendix B. The resulting transfer matrix of the complete system

\[
P^N = \begin{pmatrix}
P_{11}U_{N-1}(\xi) - U_{N-2}(\xi) & P_{12}U_{N-1}(\xi) \\
P_{21}U_{N-1}(\xi) & P_{22}U_{N-1}(\xi) - U_{N-2}(\xi)
\end{pmatrix}
\]  

(2.17)

where \( U_N(\xi) \) is the Chebychev polynomial of the second kind

\[
U_N(\xi) = \frac{\sin[(N + 1)\gamma]}{\sin \gamma}
\]  

(2.18)

where

\[
\gamma = \cos^{-1}(\xi)
\]  

(2.19)

and \( \xi \) is related to the trace of the single transfer matrix (Eqn. (2.9))

\[
\xi = \frac{1}{2} \text{Tr}(P)
\]  

(2.20)

the significance of which will be discussed subsequently.

From Eqn. (2.16), the intensity transmission and reflection coefficients can be easily
calculated. Multiplying out Eqn. (2.16) gives

\[ \Psi_0 = P_{N11} \Psi_N + P_{N12} U_N \]  
\[ U_0 = P_{N21} \Psi_N + P_{N22} U_N \]  

where \( P_{Nij} \) are the elements of \( P^N \). The waveguide connected to the right hand side of the structure is assumed to be semi-infinite (or terminated with an anechoic termination) so that no leftwards propagating wave exists in this region. The intensity transmission coefficient is the ratio of transmitted to incident intensity, which (for plane waves travelling in a waveguide) is proportional to the magnitude squared of the ratio of the transmitted to incident amplitudes. Writing Eqns. (2.21) and (2.22) in terms of amplitude coefficients of the right- and left-wards travelling waves and solving simultaneously, it can be shown that the transmission coefficient is given by

\[ \alpha_t = \left| \frac{1}{\frac{1}{2} \left[ P_{N11} + \frac{S_N}{\rho c} P_{N12} + \frac{\rho c}{S_0} P_{N21} + \frac{S_N}{S_0} P_{N22} \right]} \right|^2 \]  

where \( S_0 \) and \( S_N \) are the cross-sectional areas of the semi-infinite initial and final waveguide sections. The intensity reflection coefficient is the ratio of the reflected to incident intensity, which is again given by the magnitude squared of their amplitudes, given by

\[ \alpha_r = \left| \frac{P_{N11} + \frac{S_N}{\rho c} P_{N12} - \frac{\rho c}{S_0} P_{N21} - \frac{S_N}{S_0} P_{N22}}{P_{N11} + \frac{S_N}{\rho c} P_{N12} + \frac{\rho c}{S_0} P_{N21} + \frac{S_N}{S_0} P_{N22}} \right|^2 . \]  

Incorporating the transfer matrix coefficient values from Eqn. (2.17) allows the theoretical intensity transmission and reflection coefficients to be calculated for the periodic stack considered here. Note, if the structure under consideration is not periodic, the transmission and reflection coefficients introduced here can still be used, with the transfer matrix coefficients replaced by those for the structure under consideration.

It should be mentioned that the TMM does have some limitations, one of the most important of which is the assumption of plane wave propagation where any spherical propagation or diffraction effects are neglected. Non-linear propagation effects are also not considered. Furthermore, only continuous-wave propagation can be considered using the TMM alone. Pulse propagation can be modelled by combining the TMM with
fourier transform techniques, but this situation is more naturally treated with other techniques (for example, the finite difference time domain method). Finally, it should be noted that the TMM can be rather computationally inefficient. Although techniques were introduced above to reduce computational times for periodic systems, if modelling more complex structures with less defined periodicities, a significant number of matrix multiplications may be required to calculate the transmission properties of the structure requiring long computation times.

2.2 Bloch waves and band gaps

The quantity $\xi$ was introduced in Eqn. (2.20). It will be shown in this section that this is an important quantity associated with the development of band structure in an infinitely periodic medium.

Bloch [41] showed that the wavefunction of a conduction electron moving in a periodic potential (i.e. a crystal) is described by a Bloch wave

$$\psi_K(r) = e^{\pm iK \cdot r} u_K(r) \quad (2.25)$$

where $K$ is the Bloch wavevector, $r$ is the three-dimensional position vector and $u_K(r)$ is the Bloch function which is a function with the periodicity of the lattice

$$u_K(r + \Lambda) = u_K(r) \quad (2.26)$$

where $\Lambda$ denotes translation by a unit cell along any of the three crystal directions. For a one-dimensional system, Eqn. (2.25) reduces to

$$\psi_K(x) = e^{\pm iKx} u_K(x). \quad (2.27)$$

This is actually a general wave phenomena, and so it should be obeyed in all periodic acoustic media. Although Bloch was the first to apply this to electrons in solids, the mathematics had already been discovered by Floquet [42] and in the field of acoustics and optics, the theorem is commonly known as the Bloch-Floquet theorem.

If the medium under consideration were infinitely long and perfectly periodic, both right and left propagating waves must be Bloch waves. This can be stated (in one-
dimension) in terms of the relation between the pressure and volume velocity in the $n^{th}$ and $(n + 1)^{th}$ unit cells (after Yeh [43]) as

$$
\begin{pmatrix}
\Psi_n \\
U_n
\end{pmatrix} = e^{iK\Lambda} \begin{pmatrix}
\Psi_{n+1} \\
U_{n+1}
\end{pmatrix}.
$$

(2.28)

The coefficients are also related by Eqn. (2.8), and equating these two relations gives

$$
P \begin{pmatrix}
\Psi_n \\
U_n
\end{pmatrix} = e^{iK\Lambda} \begin{pmatrix}
\Psi_n \\
U_n
\end{pmatrix}.
$$

(2.29)

Thus, $e^{iK\Lambda}$ are the eigenvalues of $P$. Solving this yields the eigenvalue solutions

$$
e^{iK\Lambda} = \frac{P_{11} + P_{22}}{2} \pm \sqrt{\left(\frac{P_{11} + P_{22}}{2}\right)^2 + (P_{12}P_{21} - P_{11}P_{22})} \quad (2.30)
$$

but

$$\det(P) = P_{11}P_{22} - P_{12}P_{21} = 1
$$
as the transfer matrix is uni-modular giving

$$
e^{iK\Lambda} = \frac{P_{11} + P_{22}}{2} \pm \sqrt{\left(\frac{P_{11} + P_{22}}{2}\right)^2 - 1}. \quad (2.31)
$$

The two solutions can be summed to give

$$
e^{iK+\Lambda} + e^{iK-\Lambda} = e^{iK\Lambda} + e^{-iK\Lambda} = 2 \cos(K\Lambda) = P_{11} + P_{22} \quad (2.32)
$$

which can be re-arranged to give

$$
K = \frac{1}{\Lambda} \cos^{-1}\left(\frac{1}{2}(P_{11} + P_{22})\right) = \frac{1}{\Lambda} \cos^{-1} \xi \quad (2.33)
$$

where $\xi = \frac{1}{2}(P_{11} + P_{22})$ is the cosine of the Bloch phase, $K\Lambda$. Note, Eqn. (2.19) therefore defines $\gamma$ as the Bloch phase.

If $|\xi| < 1$, $K$ is entirely real (from Eqn. (2.33)) and so the Bloch waves are propagating waves. Thus, acoustic waves incident on the phononic crystal with frequencies such that $|\xi| < 1$ are able to propagate through the structure with no loss, and this therefore
corresponds to allowed energy bands in the structure. If, however, $|\xi| > 1$, $K$ is complex and the Bloch waves are evanescent (they decay exponentially with distance into the phononic crystal). Therefore frequencies such that $|\xi| > 1$ correspond to forbidden energy bands within the phononic crystal (i.e. to band gaps, regions where propagation is not allowed). Thus, the band edges can be predicted by considering the magnitude of the cosine of the Bloch phase. Indeed, by further mapping the Bloch wavenumber in terms of frequency (Eqn. (2.33)), a plot of the band structure can be built up, showing the dispersion relation of the medium and the allowed and forbidden bands. Although this analysis only holds for an infinitely periodic medium, it will be of interest to compare the results of the locally periodic media with this theoretical infinite case.

### 2.3 Modelling

The theoretical models used in this work are implemented in MATLAB. The MATLAB code is presented in Appendix D, and for a detailed description, the reader is referred to the commenting therein. This section provides a brief overview of the main concepts used in the modelling.

The main modelling is performed by the file `modulated_waveguide.m`. This model allows the user to select the type of system to model from the following list:

1. Periodic system;
2. Periodic system with a single defect;
3. Quasi-periodic system based on a Fibonacci sequence;
4. Quasi-periodic system based on a Thue-Morse sequence;
5. Quasi-periodic system based on a maximum length sequence;
6. Symmetric Fibonacci system;
7. Periodic fractal-type system;
8. Fibonacci fractal-type system;
9. Heterostructure system;
10. Pre-defined sequence of layers.

In all cases, the design frequency, number of periods, whether to use equal length perturbation and spacing layers, and whether to include mass loading can be selected. The area of the normal and perturbed sections is also defined.

For the periodic case, the transfer function elements are calculated by the function `periodic_layers_tf.m` by implementing the relations for single period transfer function elements and the Chebychev relations for multiple periods discussed in Section 2.1. When a defect is included, the transfer matrices for the periods before and after the defect are calculated in the same manner, and concatenated with the defect transfer matrix to obtain the system transfer matrix.

The situation is slightly more complicated for the other structures. For each of the quasi-periodic structures, the user selects the order of the sequence, which is generated using the functions `Fib.m`, `MLS_layer.m` and `Thue_Morse.m`. A perturbation layer is represented by a positive integer and a spacing by a negative integer where the magnitude of the integer represents the number of single layers thicknesses in each layer of the structure. The transfer matrices for each layer are then concatenated utilising this thickness information in the function `var_thick_layers_tf.m`.

If a fractal case is being considered, the spacings and areas of the perturbations are calculated based on the order of the fractal system, and the relevant transfer matrices for a single period concatenated in the function `fractal.m`. The Chebychev polynomials are then used to obtain the total transfer function for the structure. A similar process is applied for Fibonacci fractal-type systems in `fractal2.m`.

If a heterostructure is modelled, the above procedure is followed for each structure type, and the resulting transfer matrices concatenated. From the total transfer matrix for the system (single structure or heterostructure) the transmission and reflection coefficients are calculated from Eqns. (2.23) and (2.24) and then plotted.

Further calculations and plots are performed. The expected band gap positions for infinitely periodic systems with the same parameters are added to the transmission and reflection coefficient plots. For periodic systems, band structure plots and variation of band properties with varying parameters are calculated from the theoretical expressions and plotted. If defects are included, the function `fwhm.m` is used to numerically approximate the Q-factor of the peak (and warns the user if there is not sufficient resolution
to be confident in the result). If a quasi-periodic system is being considered, the auto-
correlation functions are calculated and plotted. A genetic algorithm is also used to 
minimise the transmission over a certain frequency range. The principles behind this 
are discussed in further detail in Section 7.4.
Chapter 3

Experimental configuration

This chapter presents a description of the experimental arrangement adopted in this work. The waveguide used is considered and the processes for modulating its cross-sectional area are described. The methods of characterisation of the frequency response of the waveguide structure are introduced and methods for determining the band structure considered.

3.1 The modulated waveguide construction

The basic waveguide structure used in this work is formed from sections of impedance tube (rigid rectangular plane walled metal tubes) which are connected by bolting their flanges together to form a long plane tube. The total waveguide length (from loudspeaker to measurement position) is 3 m, and the waveguide is terminated with an anechoic termination behind the measurement position to prevent reflections from the end of the tube. It was found that reflections from the source (loudspeaker) end of the waveguide (reflecting waves that were previously reflected from the modulated structure) were problematic. These were partially alleviated by inserting some loosely packed fibrous absorbent at this end of the tube, although these reflections could not be eliminated without reducing the incident sound pressure level on the modulated structure to unacceptably low levels. The experimental arrangement is shown pictorially and schematically in Fig. (3.1).

The internal dimensions of each waveguide section are square with side length $54 \pm 0.5$ mm. This places an upper limit on the range of validity of the measurements. As
the analysis assumes plane waves propagate within the tube, the valid frequency range is that for which the plane wave approximation can be assumed. For a square tube, this gives an upper limiting frequency [44] of

$$f_u = \frac{170}{d}$$  \hspace{1cm} (3.1)

where $d$ is the maximum side length of the tube (in m) which, in this case, gives an upper limiting frequency of 3150 Hz.

The cross-sectional area of the waveguide is modulated by inserting rigid aluminium
blocks into the waveguide. The blocks are square with side length 38 mm, giving a variation in area of ratio \( \eta = \frac{S_1}{S_2} = 2.02 \). Note, this gives the perturbations to the ‘normal’ waveguide as constrictions in the waveguide rather than expansions. This is due to engineering considerations, but the fundamental physics is identical to the case of expansions in the waveguide. The resulting waveguide is no longer square when the perturbations are present. This is not, however, important at the frequency ranges considered as the plane wave limit applies. The modulations in the waveguide are shown in Fig. (3.2). Vaseline is used on the surfaces of the block in contact with the waveguide structure to ensure a good seal and to avoid any absorption effects that would be associated with small gaps between the blocks and the original waveguide structure. It was found that identical results were obtained if the blocks were placed in contact with two edges as shown in Fig. (3.2) or if placed in contact with only one edge.

### 3.2 Measurement techniques

The measurements are performed using the computer based analysis software Win-MLS [45]. The system is excited using a maximum length sequence (MLS) signal and the impulse response, frequency response and phase response are determined using cross-spectral methods.
The MLS signal is discussed in general in Section 6.2.1. Its use in transfer function determination was first suggested in 1979 by Schroeder [46] and the theory and practice has since been developed and is described in detail by Rife and Vanderkooy [47]. Using cross-spectral methods, the periodic impulse response \[ h'(n) = \sum_{k=-\infty}^{\infty} h(n + kL) \] (3.2)
is determined. If the MLS sequence is longer than the impulse response of the system being measured, the periodic impulse response essentially gives the true impulse response of the system, from which the (complex) frequency response can be obtained by a fourier transform. In the absence of external noise, the periodic nature of the signal removes the need for averaging the outputs as a single period of the signal can be used to generate population statistics, removing the need for the time-consuming add-and-average technique necessary with a random noise input. In the presence of external noise, it can be shown that doubling the number of averages reduces the noise floor by 3 dB, making this an effective way to reduce the noise floor in measurements.

The system is excited with a Celestion loudspeaker amplified by a 3 W, 30 Ω power amplifier. The MLS signal, after propagating through the structure, was received by a G.R.A.S. \( \frac{1}{4} \) inch free field microphone, which has a flat frequency response (to within 1 dB) between 10 Hz and 40 kHz, and a sensitivity of 3.90 mV/Pa. This is connected to a Brüel and Kjaer pre-amplifier and a Brüel and Kjaer measuring amplifier (type 2610) before being captured by the computers sound card (Turtle Beach Santa Cruz). This completes the measurement chain.

The frequency response of the computer-amplifier-loudspeaker system is shown (as measured using WinMLS) in Fig. (3.3) together with the response of the same system when connected to the measurement waveguide system with no area modulations present. The frequency response of the computer-amplifier-loudspeaker system is not very flat (mostly due to the loudspeaker response), including a large dip at just above 1 kHz. This is rather unfortunate as the first band gap of the main system considered occurs at around 1 kHz. However, as will be shown below, by using this as a reference measurement, the response of this system is not important provided that there is sufficient output from the loudspeaker to effectively excite the system at all frequencies so
that a good signal-to-noise ratio is maintained. In order to ensure a large signal-to-noise ratio, some averaging (10 averages) of the MLS measurement is employed. Note also that the response including the plane waveguide (i.e. the waveguide structure with no perturbation blocks) follows the system response well (especially below the plane wave limit of the waveguide) with only a small reduction in level (due to small amounts of absorption that will occur along the waveguide). This indicates that the joins in the sections of impedance tube do not give rise to significant reflections as required.

3.2.1 Transmission

As discussed, cross-spectral methods are used to obtain the transfer function of the total system with the modulated structure present, $Y(\omega)$. If the system response, $X(\omega)$, is also calculated when no modulated structure is present (i.e. the response of the plane waveguide, loudspeaker, amplifier and computer), the transmission coefficient of the modulated waveguide structure can be calculated, up to an arbitrary propagation phase. Dividing the two frequency responses gives the transfer function of the modulated
structure

\[ H(\omega) = \frac{Y(\omega)}{X(\omega)}. \] (3.3)

Neglecting the propagation phase associated with the modulated structure and assuming the plane waveguide structure to be non-dispersive, this is equal to the amplitude transmission coefficient, which gives an intensity transmission coefficient

\[ \alpha_t = \left| \frac{Y(\omega)}{X(\omega)} \right|^2. \] (3.4)

### 3.2.2 Band structure

The band structure is slightly more complicated to calculate. The relative phase delay introduced by the modulated waveguide structure (with respect to the plane, unperturbed waveguide) can be obtained by taking the phase of the transfer function relating the frequency response at the end of the structure with and without the modulated waveguide present (Eqn. (3.3))

\[ \phi(\omega) = \arg \left( \frac{Y(\omega)}{X(\omega)} \right). \] (3.5)

This phase delay can be used to calculate a phase velocity [16]

\[ v(\omega) = \frac{c}{c\phi(\omega) - \frac{\omega}{\omega L} + 1} \] (3.6)

where \( c \) is the speed of sound in the unmodulated waveguide and \( L \) is the total length of the periodic array. The phase velocity is related to the wavenumber

\[ v(\omega) = \omega \frac{k}{k} \] (3.7)

which can be rearranged to obtain the wavevector

\[ k(\omega) = \frac{\omega}{v(\omega)}. \] (3.8)

This corresponds to the Bloch wavevector in the structure.
Chapter 4

Periodic systems

Initially, the simplest case of a locally periodic system will be considered. This chapter presents the results of a theoretical and experimental investigation into such structures and the discussions of these results. The effects of varying the number of periods in the structure will be considered, as will the effects of varying the magnitude of the modulation of the waveguide (i.e. the difference in cross-sectional area between the sections of waveguide). The effects on the transmission coefficient will be presented. The results will also be presented in terms of a band structure analysis, familiar from solid state physics.

4.1 Theoretical band structure

Although not strictly valid for locally periodic media, much insight can be gained by analysing the system using the theory of Bloch waves introduced in Section 2.2. As the number of periods in the locally periodic structure is increased, the band structure of the locally periodic structure would be expected to tend towards that of the infinitely periodic structure, thus allowing the effective prediction of band gaps.

From Eqns. (2.20) and (2.9), the cosine of the Bloch phase is given by

\[ \xi = \frac{1}{2} \left( 2 \cos(2ka) \cos(k\ell) - \frac{S_1}{S_2} + \frac{S_2}{S_1} \right) \sin(2ka) \sin(k\ell) \]  

(4.1)

where \( S_1 \) and \( S_2 \) are the cross-sectional areas of the ‘normal’ and perturbed waveguide sections respectively, \( 2a \) is the length of a single perturbation, \( \ell \) is the length of the
spacing between perturbations and $k = \frac{\omega}{c}$ is the wavenumber. Letting

$$\eta = \frac{S_1}{S_2}$$

(4.2)

and defining

$$\varepsilon_+ = \frac{1}{2} \left[ \eta + \frac{1}{\eta} \right] = \frac{1}{2} \left[ \frac{S_1}{S_2} + \frac{S_2}{S_1} \right]$$

(4.3)

Eqn. (4.1) reduces to

$$\xi = \cos(2ka) \cos(k\ell) - \varepsilon_+ \sin(2ka) \sin(k\ell).$$

(4.4)

As discussed, when $|\xi| > 1$, the Bloch waves are evanescent, and so these frequencies correspond to band gaps of the structure. Furthermore, the full band structure (dispersion relation relating the Bloch wavenumber and frequency) can be calculated from Eqn. (2.33).

It is of interest to consider the specific case where the period $\Lambda = 4a$ (i.e. the length of the perturbations to the waveguide is equal to the length of the ‘normal’ sections, giving $\ell = 2a$). In this case, Eqn. (4.4) reduces to

$$\xi = 1 - (1 + \varepsilon_+) \sin^2(2ka).$$

(4.5)

An interesting feature of this equation is that $0 \leq \sin^2(2ka) \leq 1$ and so

$$\xi \leq 1 \quad \forall \quad ka.$$ 

(4.6)

Thus, $\xi > 1$ is not possible, and so the only regions where propagation of acoustic waves is forbidden is at frequencies such that $\xi < -1$. It will be seen that this corresponds to the situation where band gaps only open up at the so-called Brillouin zone (BZ) edge and not the centre.

**Aside: The reciprocal lattice and Brillouin zones**

The reciprocal lattice is a concept in wide use in the description of crystal systems, where the lattice is represented in reciprocal (or $k$-) space rather than real space. Note, the reciprocal lattice is a mathematical abstraction useful in the description of crystal
systems – it does not physically exist. The basis of the reciprocal lattice is defined by the set of unit vectors normal to the principle planes of the crystal structure. In the current case, as the original lattice of the underlying structure is only defined in one dimension, the reciprocal lattice will only be defined in one dimension also (i.e. will consist of planes). These planes will be normal to the planes of the original lattice structure, and the spacing of the planes in the reciprocal lattice are inversely related to those of the original lattice

\[ d^* = \frac{2\pi}{d} \]  

(4.7)

where \( d^* \) represents the interplanar spacing of the reciprocal lattice and \( d \) of the equivalent real-space lattice.

The primitive unit cell of the reciprocal lattice is termed the Brillouin zone, BZ, (technically the first BZ) of the reciprocal lattice, which is found, in one-dimension, by bisecting the line joining adjacent reciprocal lattice points. The plane so-formed by this bisection is a Bragg plane, i.e. a plane that satisfies the Bragg condition for reflection. In the current case, the BZ is therefore defined over the region

\[ -\frac{\pi}{\Lambda} \leq K \leq \frac{\pi}{\Lambda} \]  

(4.8)

where \( \Lambda \) is the period of the real-space lattice and \( K \) is the wavevector (the Bloch wavevector). Therefore, the centre of the BZ is defined at \( K = 0 \) and the edge of the BZ at \( K = \pm \frac{\pi}{\Lambda} \).

The BZ is an important concept due to the consequences of Bloch’s theorem. Due to the periodicity of the lattice, the Bloch wavevector is only unique up to a reciprocal lattice vector. Therefore, all wave-states are contained within the first BZ and simply repeated in subsequent BZs. Therefore, the crystal can be completely characterised by considering the Bloch wavevectors within the first BZ, greatly simplifying the analysis of periodic systems.

It is possible to predict the frequencies at the edge of the band gap for this simple one-dimensional system by solving Eqn. (4.5) for \( \xi = -1 \) giving

\[ 2ka = \sin^{-1}\left(\sqrt{\frac{2}{1 + \varepsilon_+}}\right). \]  

(4.9)
Substituting $k = \frac{2\pi f}{c}$ and re-arranging gives the lowest band edge frequency

$$f_{l_1} = \frac{c}{4\pi a} \sin^{-1} \left( \sqrt{\frac{2}{1 + \varepsilon_+}} \right).$$ (4.10)

Due to the properties of the sinusoid function, the equality $\xi = -1$ is next satisfied if $2ka \rightarrow \pi - 2ka$ giving the upper edge of the lowest band gap

$$f_{u_1} = \frac{c}{4\pi a} \left[ \pi - \sin^{-1} \left( \sqrt{\frac{2}{1 + \varepsilon_+}} \right) \right].$$ (4.11)

Thus, the width of the first band gap can be calculated

$$\Delta f = f_{u_1} - f_{l_1} = \frac{c}{4\pi a} \left[ \pi - 2 \sin^{-1} \left( \sqrt{\frac{2}{1 + \varepsilon_+}} \right) \right]$$ (4.12)

and substituting for $\varepsilon_+$ from Eqn. (4.3) gives the bandwidth

$$\Delta f = f_{u_1} - f_{l_1} = \frac{c}{4\pi a} \left[ \pi - 2 \sin^{-1} \left( \frac{2\sqrt{\eta}}{\eta + 1} \right) \right]$$ (4.13)

where $\eta$ was defined in Eqn. (4.2). Besides physical constants, the width of the band gap can be seen to have a dependence on the length of the perturbation sections and on the ratio between the two areas of the waveguide system. These effects will be considered in further detail subsequently.

This analysis has been presented for the first band gap in the system. However, due to the properties of the sinusoidal function, the subsequent lower and upper band edge frequencies are related to the first lower and upper edge respectively by consecutive integer multiples of $\pi$, and so the width of the higher order band gaps is equal to the first band gap described here. Note, this only holds in the current case when the perturbation length is equal to half of the overall period. The band structure will be considered in relation to the transmission properties of the medium below.

4.2 Theoretical results for periodic systems

Initially, the theoretical results from modelling various locally periodic systems will be considered. This allows the fundamental features of the systems to be identified. The
results will subsequently be compared with experimental data.

### 4.2.1 A Bragg stack

Consider initially a Bragg stack which, in the present case, is formed from alternating layers of large and small cross-sectional area. The thickness of each layer is equal. This is represented schematically by Fig. (2.2) where the period \( \Lambda = 4a \) and hence the spacing between the perturbations, \( \ell = 2a \) which is equal to the length of the perturbations themselves.

Consider a structure with a period length of 0.172 m formed in a square waveguide with open areas of 0.054\(^2\) m\(^2\) for the unperturbed sections and 0.038\(^2\) m\(^2\) for the perturbed sections. This will be denoted the *standard test arrangement* for this chapter. These values are used to allow easy comparison with the experimental data below. The predicted transmission coefficient is shown for a 10 period structure in Fig. (4.1). The results are only shown up to a frequency of 3 kHz, as this is approximately the plane wave limit for a waveguide of this cross-section, and hence the upper end of the range of validity of the model.

As can be clearly seen, the structure exhibits well defined stop bands with a series of

![Figure 4.1](image_url)

**Figure 4.1:** The intensity transmission coefficient for a 10 period Bragg stack constructed in the standard test arrangement. The theoretical positions of the stop bands for an infinite periodic medium are also shown (red dotted line). Clear band gaps are present in the transmission spectrum that show good agreement with the expected band gaps from an infinitely periodic medium.
peaks and dips of varying magnitude between the stop bands, which are characteristic of interference phenomena. It can also be seen that the stop bands are very well described by those resulting from an infinite medium for which the theory presented in Section 4.1 applies and it is rather surprising that the locally periodic structure exhibits such well defined stop bands with only a relatively small number of periods.

It is of interest to consider the location in frequency of the first stop band, or band gap. As the infinite case has been shown to have a band gap which is very closely represented by the locally periodic case with 10 periods, the analysis presented in Section 4.1 can be used to calculate the centre frequency of the band gap. From Eqns. (4.10) and (4.12), the centre frequency is given by

\[
    f_{c_1} = f_{l_1} + \frac{\Delta f}{2} = \frac{c}{4\pi a} \sin^{-1} \left( \sqrt{\frac{2}{1 + \varepsilon_+}} \right) + \frac{1}{2} \frac{c}{4\pi a} \left[ \pi - 2 \sin^{-1} \left( \sqrt{\frac{2}{1 + \varepsilon_+}} \right) \right] = \frac{c}{8a} = \frac{c}{2\Lambda}. \tag{4.14}
\]

In the present case, this gives \( f_c \approx 1 \text{ kHz} \) as seen in Fig. (4.1).

It is useful to give this theoretical value a qualitative justification. Consider the waves reflected at each period through the structure, shown schematically in Fig. (4.2) for a small number of periods. Fig. (4.2(a)) represents the case for sound incident on the structure with a frequency equal to the centre of the band gap. At each discontinuity in the area of the structure, the incident wave is partially reflected. If the reflected waves are in phase with each other, as is the case here, a standing wave will be formed, and sound will not propagate through the material leading to the formation of a band gap as seen. The frequency at which the reflected waves are exactly in phase is the centre frequency of the band gap. To satisfy this condition, the period distance must be an odd integer number of half-wavelengths

\[
    \Lambda = m \frac{\lambda}{2} \tag{4.15}
\]

where \( m = 1, 3, 5, \ldots \) giving a centre frequency

\[
    f_c = \frac{c}{\Lambda} = \frac{c}{2\Lambda}. \tag{4.16}
\]
Figure 4.2: A schematic representation of the waves partially reflected at each period of the structure (a) for a frequency at the centre of the band gap and (b) for a frequency outside of the band gap. In (a) the reflected waves from each period are in phase with each other, leading to the formation of a band gap. In (b), however, the reflected waves are out of phase with each other, and so incident sound at this frequency can propagate through the structure.

for the first band gap as in Eqn. (4.14). This justifies both the formation of a band gap and also its centre frequency, and explains the formation of the next band gap at a frequency three times that of the first, and not double. In the current case, the perturbed layers are the same thickness as the ‘normal’ layers of waveguide, and so each layer must have a thickness of one-quarter wavelength for the first band gap. This corresponds to the optical thickness of the layers in the well known optical Bragg stack [25].

Consider now the case where the period length is not close to a half-wavelength of the incident wave, shown in Fig. (4.2(b)). The partial reflections of the incident wave still occur at the discontinuities in the waveguide structure, but now the reflected waves are not in phase, as shown. The degree of constructive or destructive interference of the reflected waves will depend on the exact wavelength of the incident wave with respect to the period length, leading to the interference pattern seen in Fig. (4.1). Crucially, however, a standing wave is not formed, and so sound at these frequencies can propagate through the structure (with some attenuation depending on the frequency).

The fundamental properties of the structure can be represented by its band structure which maps the allowed wavevectors of the Bloch wavefunction that correspond to travelling (rather than evanescent) waves against frequency (i.e. the dispersion relation of
the real component of the Bloch wavevector). For transmission of an acoustic wave, its frequency must be such that there is an allowed propagating wavestate, which is conveniently represented by its band structure. This is shown in Fig. (4.3(a)) in the reduced zone scheme, where all dispersion curves are represented within the first Brillouin zone (BZ). This is valid due to Bloch’s theorem (see the discussion on BZs above). Note, the band structure is strictly only valid for the infinitely periodic system, but the band gaps have been seen to be well approximated by the 10 period system considered here.

Regions of frequency (for example around 1 kHz) can be clearly seen where there is no allowed solution of the Bloch wavevector (a gap is seen between the allowed states). This corresponds to the band gaps of the structure, and comparison with Fig. (4.1) shows that these gaps occur over the same frequency range as the transmission stop band. In these regions, the Bloch wavenumber is complex, leading to an evanescent wave and hence zero transmission (for the infinite structure). Thus, the dispersion relation of the structure can be seen to be separated into allowed bands separated by forbidden regions, or band gaps.

An interesting feature of this system is that band gaps open up only at the edge of the BZ, and not at the centre. It was stated above that band gaps only occur at the edge of the BZ if the cosine of the Bloch phase, $\xi$, is never greater than unity. This

![Figure 4.3](image-url)

**Figure 4.3:** (a) The band structure and (b) the cosine of the Bloch phase for an infinite Bragg stack constructed in the standard test arrangement. The reduced zone scheme is used, where all dispersion curves have been folded into the first BZ. Clear band gaps are seen which occur at the edge of the BZ, coinciding with frequencies such that the cosine of the Bloch phase satisfies $\xi < -1$. 
can be seen to occur in this case, as shown in Fig. (4.3(b)). The band gaps can be seen to correspond to frequencies where $|\xi| > 1$, as expected from the above theoretical discussion. However, $\xi \leq 1$ for all frequencies, and so band gaps only occur at the edge of the BZ. Alternative situations where band gaps can open up at the centre of the BZ in addition to the edge will be considered below.

In addition to controlling the position of the band gaps, the period length of the structure also has an effect on the width of the band gaps. Eqn. (4.13) shows that the band gap width (in frequency) is inversely proportional to the period. Thus, the band gaps can be formed at lower frequencies by increasing the period length, however this results in narrower band gaps. Conversely, wide band gaps can be created by decreasing the period length, but the band gaps will occur at higher frequencies. This is represented by the band structure for two different period lengths in Fig. (4.4), supporting the above discussion.

### 4.2.2 Varying the number of periods

The band structures presented here are only strictly valid when considering an infinite medium with total periodicity. It was shown that for 10 periods of the structure, good

![Figure 4.4: The band structure for an infinite Bragg stack with a period length of 0.086 m (blue line) and 0.344 m (red line) with area modulations of the standard test arrangement. The reduced zone scheme is used. It can be seen that longer period lengths give rise to band gaps occurring at lower frequencies, but these band gaps are also narrower than those resulting from structures with shorter periods.](image)
agreement of the band gaps in the transmission spectrum resulted but it is of interest to consider the behaviour of the system as the number of periods is varied. The transmission coefficient for the modulated waveguide in the standard test arrangement is shown in Fig. (4.5) for 1, 2, 5, 8, 10 and 30 periods of the structure.

Consider initially the case of a single period, which corresponds to a single expansion or contraction of length $2a = 0.086\, \text{m}$ in an infinity long waveguide. At very low frequencies (such that $2ka \ll 1$), this system is often treated using a lumped-parameter approach, where it can be easily shown (see, for example, Kinsler et al. [4]) that this results in a low pass filter. This effect is indeed seen here at low frequencies (below approximately 500 Hz). At higher frequencies, however, the transmission increases, before this pattern is repeated. Therefore, the transmission shows distinct peaks and dips, with the minimum in transmission coincident with the centre of the theoretical band gap centre frequency for an infinite number of periods in the medium. Thus, even with a single period, this system exhibits the pre-cursors of band structure.

As the number of periods is increased, the dips in transmission at the band gap frequencies can be seen to deepen, whilst the peaks develop more structure, due to the increased number of layers leading to a more complex interference of incident and reflected waves outside of the band gap frequencies. Note, the transmission coefficient can be seen to be periodic in frequency, with a period (in frequency) of twice the first band gap centre frequency. Furthermore, each period of the transmission coefficient can be seen to be symmetrical about the band gap centre frequency.

When 5 periods of the structure exist, the transmission at the centre frequency of the theoretical band gap can be seen to be very close to zero transmission, although the transmission begins to increase within the frequency range of the theoretical band gap. As the number of periods is increased further, the transmission in the frequency range of the theoretical band gap reduces (i.e. the dip in transmission becomes broader and flatter) and the slope of the transmission coefficient on either side of the dip becomes sharper. The transmission coefficient also shows more peaks and dips in frequencies outside of the band gap range due to the more complicated sequence of reflections from the multiple layers. Above approximately 8 periods, the theoretical band gap for the infinity periodic structure is very well approximated by the locally periodic structure, with a flat stop band of virtually zero transmission and very steep roll-off from the pass-
Figure 4.5: The intensity transmission coefficient for a Bragg stack constructed in the standard test arrangement for 1, 2, 5, 8, 10 and 30 periods of structure. The theoretical positions of the stop bands for an infinite periodic medium are also shown (red dotted line). As the number of periods is increased, the dips in transmission are seen to deepen and flatten (tending towards those of the infinitely periodic structure), indicating the formation of full band gaps. The transmission between the band gaps also becomes more complex with larger numbers of peaks and dips due to an increased number of interference paths in longer structures.
bands. Increasing the number of periods above this number does increase the steepness of the band edges slightly, but mainly serves to introduce more peaks and dips into the transmission between the band gaps.

Thus, in the language of acoustical filters, the structure forms a band-stop filter with a very steep roll-off and flat reject-band. As an aside, it is of interest to compare the behaviour of the waveguide transmission with that which could be obtained from a digital filter design. As an example, a digital Butterworth filter has been generated in MATLAB using the *butter* design tool. The -3 dB points of the Butterworth filter have been specified to be the same as for the Bragg stack. The stop bands are shown, for various orders, in Fig. (4.6). Clearly, the modulated waveguide achieves steeper roll-off rates than are reasonably achievable with a digital filter (indeed, even a 32\textsuperscript{nd} order filter cannot produce such a steep roll-off). Of course, the Butterworth design achieves a much flatter pass-band region. This example, however, serves to illustrate the ability of the modulated waveguide system to achieve very sharp cut-off with little expense.
4.2.3 Varying the area ratio

Consider the effects of varying the ratio of the areas in the perturbed and ‘normal’ sections of waveguide, \( \eta = \frac{S_1}{S_2} \). The case considered above (Fig. (4.1)) has an area ratio of \( \eta \approx 2 \), i.e. the cross-sectional area of the perturbed sections is half that of the ‘normal’ sections of waveguide. Consider if the ratio is reduced to \( \eta = 1.35 \) by increasing the open area of the perturbed sections of waveguide to 0.047 m\(^2\). The resulting transmission coefficient is shown in Fig. (4.7).

Comparison with Fig. (4.1) shows that the transmission minima are located at the same frequencies as for the case with different \( \eta \) as would be expected. However, the transmission minima can be seen to be much narrower and with less-steep roll-off and higher transmission in the stop band than the equivalent system with larger \( \eta \) (i.e. the stop-bands are less well defined, only representing pre-cursors of full band gaps).

As \( \eta \) becomes smaller (closer to unity) the areas of the perturbed and unperturbed sections of the waveguide tend towards each other. Therefore, the impedance discontinuity at each change of area becomes smaller. As the impedance discontinuity is smaller,

![Figure 4.7:](image-url)

**Figure 4.7:** The intensity transmission coefficient for a 10 period Bragg stack with a period length of 0.172 m formed in a square waveguide with open areas of 0.054 m\(^2\) for the unperturbed sections and 0.047 m\(^2\) for the perturbed sections giving an area ratio \( \eta = 1.35 \). The theoretical positions of the stop bands for an infinite periodic medium are also shown (red dotted line). The band gaps are seen to be narrower than for systems with a higher value of \( \eta \) and to be less fully formed in a 10 period structure due to the smaller impedance mismatch between the sections of waveguide.
the component of the wave reflected will also be reduced. With a smaller reflected component, the sum of the in-phase components of the rear reflected waves will also be smaller, and so for a given number of periods, the band gaps will be less completely formed, as is seen here. Increasing the number of periods would provide a larger number of reflections, and so it is still possible to form a well defined band gap with sharp edges as seen in Fig. (4.1), but this now requires a larger number of periods.

The smaller impedance discontinuities also qualitatively describes the width variation of the band gap. Even for an infinitely periodic structure, the band gap width can be seen to be reduced for smaller $\eta$ (compare the theoretical band gap locations in Figs. (4.1) and (4.7)). This can be treated quantitatively by considering the expression for the band gap width derived above (Eqn. (4.12)).

The factor with the dependence on $\eta$ is shown in Fig. (4.8(a)). Clearly, this factor shows a large variation with $\eta$. For a large band gap, the factor must be small, requiring a large value of $\eta$. Note, as $\eta \to 1$, the factor $\left(\sin^{-1}\left(\frac{2\sqrt{\eta}}{\eta+1}\right)\right) \to \frac{\pi}{2}$ and so (from Eqn. (4.13)) the band gap width tends to zero. This is as expected, as $\eta = 1$ corresponds to $S_1 = S_2$ and hence no change in area along the waveguide and therefore no reflections, meaning a band gap does not form. The variation of actual band gap width with $\eta$ for an infinitely periodic Bragg stack with period $\Lambda = 0.172$ m is shown in Fig. (4.8(b)). This shows the expected trend, with no band gap at $\eta = 1$. As $\eta \to \infty$, the band gap width tends to a limiting value equal to twice the centre frequency of the first band gap.

**Figure 4.8:** (a) The variation of $\left(\sin^{-1}\left(\frac{2\sqrt{\eta}}{\eta+1}\right)\right)$ with $\eta$ (for $\eta > 1$) and (b) the corresponding variation in theoretical band gap width with $\eta$ for a Bragg stack with a period length of 0.172 m.
4.2.4 Non-equal perturbation and spacing lengths

Consider a structure where the perturbation and spacing layers are not of equal length. As an example, the transmission coefficient (and schematic representation of the structure) of a 10 period structure where the spacing length is double that of the perturbation layer thickness is shown in Fig. (4.9). As before, the perturbation layers are taken to have a length of 0.086 m giving a total period length of 0.258 m, and the ratio of areas in the perturbed and ‘normal’ sections is as in the standard test arrangement. The lowest band gap can be seen to occur centred on approximately 665 Hz, consistent with the period length and the above discussion. However, instead of the transmission coefficient being symmetrical about this band gap, it can be seen to be a member of a doublet of gaps.

![Schematic representation of the waveguide and intensity transmission coefficient](image1)

**Figure 4.9:** A schematic representation of the waveguide (top) and the intensity transmission coefficient (bottom) for a 10 period modulated waveguide structure with a period length of 0.258 m where the perturbations are spaced at twice their length. The ratio of areas in the perturbed and ‘normal’ sections is as in the standard test arrangement. The theoretical positions of the stop bands for an infinite periodic medium are also shown (red dotted line). Comparing the transmission of this structure to that of the Bragg stack in Fig. (4.1) shows that the band gaps have been split into two, spaced about the original band gap positions.
This can be seen to result from a second gap opening up at the centre of the BZ. Consider the band structure and the cosine of the Bloch phase shown in Fig. (4.10). The first band gap occurs at the edge of the BZ when the cosine of the Bloch phase satisfies $\xi < -1$ as for the Bragg stack. However, the cosine of the Bloch phase can now take values such that $\xi > 1$ leading to the second band gap opening up at the centre of the BZ. Note also, the combination of layers is such that the next potential band gap at the edge of the BZ does not form, and so the next band gap that results also forms at the centre of the BZ.

The resulting transmission coefficient can be viewed as being derived from the original Bragg stack transmission but with each of the original stop bands split into two and spaced symmetrically about the original band gap centre frequency. This trend continues for larger spacings. For example, by increasing the spacing of the perturbations to three times their length, the resulting transmission stop bands are split into three. By utilising different spacings that are not integer multiples of the perturbation length, more complex transmission patterns are produced and their symmetry can be destroyed. For example, the transmission coefficient for a 10 period structure with a spacing of 0.05 m is shown in Fig. (4.11). Clearly, any symmetry in the transmission co-

![Figure 4.10](image_url)

**Figure 4.10:** (a) The band structure and (b) the cosine of the Bloch phase for an infinitely periodic modulated waveguide structure with a period length of 0.258 m where the perturbations are spaced at twice their length. The ratio of areas in the perturbed and ‘normal’ sections is as in the standard test arrangement. The reduced zone scheme is used, where all dispersion curves have been folded into the first BZ. Band gaps are now seen to occur both at the edge of the BZ (where $\xi < -1$) as was seen with the Bragg stack but also at the centre of the BZ which occurs when $\xi > 1$. 

4.3 Experimental investigation of modelling results

This section presents the results of an experimental investigation into the periodic structures of the kind discussed above. The experimental techniques used to make these measurements were discussed in Chapter 3.
4.3.1 A Bragg stack

Consider a Bragg stack constructed in the standard test arrangement discussed in Section 4.2.1, with a period length of 0.172 m formed in a waveguide with areas of 0.054 m$^2$ for the unperturbed sections and 0.038 m$^2$ for the perturbed sections. The measured transmission coefficient of an 8 period system is shown in Fig. (4.12) along with the theoretical calculations as discussed in Section 4.2.

Above approximately 2.9 kHz, the agreement between theory and experiment is very poor. However, from Eqn. (3.1), this frequency is approaching that at which the plane wave limit no longer holds in the waveguide. Therefore, the disagreement at these frequencies is likely due to non-plane wave propagation (especially at the changes in cross-sectional area), in which case the assumptions of the theory are no longer applicable and the model breaks down. The measurement results will therefore only be considered up to this frequency. Below this frequency, the general trends of the measured results are predicted by the theory, although the features (both transmission

![Figure 4.12](image_url)

**Figure 4.12:** The transmission coefficient for an 8 period Bragg stack constructed in the standard test arrangement. Experimentally measured data for such a system is shown as are the theoretical predictions discussed in Section 4.2. The general trends of the measured results are reproduced by the theory, although the features occur at slightly too high frequencies. The observed dip at $\sim 2$ kHz is also not predicted by the theory. Agreement above $\sim 2.9$ kHz is poor due to the failure of the plane wave assumption in the model at high frequencies.
peaks and dips and band gaps) occur at slightly too high frequencies and the observed dip at $\sim 2$ kHz is also not predicted by theoretical calculations. The first point suggests that the thicknesses and spacings of the blocks that were placed in the waveguide to modulate the area were slightly longer than the values used to model the data, thus resulting in band gaps at slightly lower frequencies than expected. Note, this could also be explained by variations in the speed of sound. However, the laboratory was air conditioned to a temperature of $20^\circ$C, and so confidence can be placed in the value of $c = 343$ ms$^{-1}$ used in the modelling. The dip at approximately 2 kHz suggests that some periodicity existed within the structure with a period half that of the main structure periodicity, thus leading to a dip in transmission at this frequency due to constructive interference of reflected waves by a structure with this new periodicity.

In the above modelling, the radiation impedance (mass loading) of the structure has been neglected. It will be shown that including the mass loading effects can account for both of the above discrepancies between the theory and the measured data. When there is a change in cross-sectional area, a radiation impedance exists. In a lumped-element approximation, at the change in area, the air acts as a plane piston, and the radiation impedance acts on this piston. Although initially neglected in this model as it is normally a small effect, it appears that the periodicity of the structure, resulting in a radiation impedance acting every half period of the structure (i.e. at every area discontinuity) leads to much more noticeable effects. The radiation impedance can be included into the model by concatenating a transfer matrix

$$P_{rad} = \begin{pmatrix} 1 & Z_{A_{rad}} \\ 0 & 1 \end{pmatrix}$$

(4.17)

(where $Z_{A_{rad}}$ is the radiation impedance in acoustic units) before and after the transfer matrix describing each perturbation of the waveguide.

Consider the experimental arrangement utilized here. A cross-section of the waveguide when an aluminium block is inserted (to cause a perturbation in the cross-sectional area of the waveguide) is shown schematically in Fig. (4.13(a)). This situation was shown pictorially in Fig. (3.2). At the change in area of the waveguide (at the ends of each perturbation block) it may be thought that the remaining L-shaped area of the waveguide acts as the piston of air in the lumped element model for the calculation of the radia-
Figure 4.13: (a) A cross-section of the waveguide when an aluminium block is inserted (to cause a perturbation in the cross-sectional area of the waveguide) and (b) the image sources created by reflection in the walls of the waveguide. The cross-shaped area corresponding to the ‘piston of air’ in the lumped element model is outlined in red.

It is non-trivial to calculate the exact radiation impedance for this arrangement and indeed it is beyond the scope of this work. As an approximation the case of a plane circular piston in an infinite baffle will be considered (as analytical solutions exist for this case), with the area of the modelled circular piston equal to the area of the equivalent cross-shaped section in the experimental arrangement. The radiation impedance of a plane circular piston in an infinite baffle is discussed in Appendix C and (at low frequencies and neglecting the small resistive component) is given (in acoustic units) by

\[ Z_{A_{rad}} \approx j \frac{\rho_0 c}{S_{ml}} \frac{8}{3\pi} (ka_{ml}) \]  

(4.18)

where \( S_{ml} \) and \( a_{ml} \) are the area and radius respectively of the equivalent circular piston used in the approximation. In the current case, this gives a radiation impedance

\[ Z_{A_{rad}} \approx 7.6j\omega. \]  

(4.19)

This is equivalent to an added mass in a lumped element model, and so is termed mass loading. The measured results and theoretical predictions incorporating this mass
loading are shown in Fig. (4.14).

With this mass loading included, the measured results are not modelled that accurately. However, a new band gap is seen to form at approximately the correct frequency to account for the extra observed dip in transmission, suggesting that the mass loading should be included, but currently has too large an effect. By reducing the magnitude of the radiation impedance to give

$$Z_{A_{rad}} = 4j\omega$$  \hspace{1cm} (4.20)

the theoretical results agree very closely with the experimentally measured data, both in terms of band gaps and transmission peaks and dips. The close agreement of the theoretical (including the smaller radiation impedance) and measured results shows that mass loading is sufficient to account for the previous discrepancies in the model.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure4_14.png}
\caption{The transmission coefficient for an 8 period Bragg stack constructed in the standard test arrangement. Experimentally measured data for such a system is shown with the theoretical predictions incorporating mass loading, where two values for the mass loading have been used. When the mass loading is modelled as that acting on a plane circular piston of the same size as the change in area in the structure, the measured results are not well modelled. However, reducing the magnitude of the radiation impedance leads to theoretical results that agree very closely with the experimentally measured data.}
\end{figure}
Note, this value for the radiation impedance is derived from an empirical fit to the experimental data. However, in the absence of a derived quantity for the radiation impedance of the exact experimental arrangement, this shall be assumed to be sufficient to accurately describe the situation and this expression for the radiation impedance shall be used for the remainder of this work when mass loading is included.

The difference in magnitude between the two radiation impedances used above (the derived and empirical values) is likely due to two principal factors. Firstly, the piston on which the radiation impedance acts (in the lumped element analogy) is not circular, as was assumed in the calculation above. This will necessarily lead to some discrepancies in the final value. Secondly, and perhaps more importantly, the analysis above assumes radiation into $2\pi$ space, whereas in the experimental arrangement used here, the radiation is into a tube with dimensions that are not significantly larger than the dimensions of the tube when the perturbation blocks are present. Assuming a low frequency limit (where lumped parameter models apply) it is possible to account for this (approximately) by subtracting the previously calculated radiation impedance for the cross-shaped piston within the tube from the radiation impedance of the entire waveguide (the total area of the waveguide shown in Fig. (4.13(b)) including all image sources) radiating as a piston into $2\pi$ space. The latter radiation impedance can be calculated from Eqn. (4.18) and takes the value

$$ (Z_{A_{rad}})_{waveguide} = 10.7 j \omega $$

and subtracting the calculated radiation impedance of the cross-shaped piston (Eqn. 4.19) gives an approximate radiation impedance acting at each change in area of the waveguide of

$$ (Z_{A_{rad}}) \approx 3.1 j \omega. $$

This is much closer to the empirical value (Eqn. (4.20)) required for accurate modelling of the measured transmission, suggesting this empirical value is reasonable. Finally, it should be noted that the radiation impedance acting at each area perturbation of the waveguide will be further modified as the area perturbations occur in an array rather than in isolation.

Therefore, it has been shown that the mass loading accounts for the presence of the extra large dip in transmission not previously seen in the modelling. It also accounts for the observed frequency shift between the measured results and the original theor-
ical predictions discussed above. It was stated above that these could be accounted for by increasing the length of the waveguide perturbations slightly. This increase in length is incorporated in the mass loading as the radiation impedance can be viewed (in the lumped-element approximation) as an increase in length due to the increase in mass. These so-called end corrections are well known (for example in predicting the resonant frequencies of Helmholtz resonators [4]) and can be seen in this case to result in predictions of the band gap frequencies and peaks and dips in the transmission of the structure that agree very well with the theoretical results. Note, the magnitude of the transmission peaks are, in general, somewhat lower than those predicted by the theory. This is most likely due to a small amount of absorption within the waveguide structure which has not been included within the model. These effects are especially prevalent at higher frequencies, as would be expected.

Good agreement between the experimental data and the theoretical predictions (including mass loading) is seen for other numbers of periods of the structure. However, for small numbers of periods, the transmission coefficients display some small oscillatory behaviour not theoretically present and not observed with larger numbers of periods. An example, for a 3 period modulated waveguide, is shown in Fig. (4.15).

The general agreement between the theoretical and experimental results is good, but some extra oscillations are indeed seen in the measured results not present in the theoretical predictions. It was found experimentally that this was due to reflections from the loudspeaker at the source end of the waveguide. These were partially alleviated by placing some loosely packed absorbent (fiberglass wool) at the source end of the waveguide and this reduced these extra oscillations to the degree where they are not visible for higher numbers of periods. For low numbers of periods, however, the problem still persisted, and could not be further reduced without detrimentally attenuating the sound levels incident on the modulated waveguide structure. The extra oscillations seen in the transmission coefficients of modulated waveguides with a low number of periods can therefore be judged to be due to experimental error, and not theoretical inaccuracies.

### 4.3.2 Band structure

Utilising the theory of Section 3.2.2, it is possible to obtain an experimental measure of the band structure of the modulated waveguide. This is shown in Fig. (4.16) for an
Figure 4.15: The transmission coefficient for a 3 period Bragg stack constructed in the standard test arrangement. Experimentally measured data is shown with the theoretical predictions incorporating mass loading. The general agreement is good, but some extra oscillations are seen in the measured results not present in the theoretical predictions.

8 period Bragg stack constructed in the standard test arrangement. The general form of the measured data can be seen to be predicted by the theoretical calculations, but the scaling is slightly in error, with the theoretical and experimental results diverging in magnitude particularly at higher frequencies. The probable cause of this divergence is experimental error. The phase of the measurement performed is very sensitive to microphone placement, particularly at higher frequencies where the wavelength is shorter. The phase delay is calculated from the relative phase difference between the measured response with the Bragg stack structure present and a reference measurement for the plane waveguide, and variation in, for example, microphone placement for these two measurements could lead to a slight increase or decrease of phase delay over the desired value (due to a slight increase or decrease in the propagation phase to the microphone). Although care was taken to place the microphone in the same place for each measurement, this was only accurate to approximately ±2 mm. This gives a potential error in
Figure 4.16: The band structure (real part of the Bloch phase, shown for $K > 0$ only) for an 8 period Bragg stack constructed in the standard test arrangement. Experimentally measured data is shown with theoretical predictions for an infinite medium. The general form of the measured data is predicted by the theoretical calculations, but the scaling can be seen to be slightly in error.

The phase velocity (using Eqn. (3.6))

$$\delta v(\omega) = \frac{c_0}{\omega L} \left( \frac{\bar{c}_0}{c_0(\phi(\omega) \pm \delta \phi(\omega))} + 1 \right) - \frac{c_0}{\omega L} \left( \frac{\bar{c}_0 \phi(\omega)}{\omega L} + 1 \right)$$

(4.23)

where $\delta \phi(\omega)$ is the error in the phase delay due to increased or decreased propagation phase

$$\delta \phi(\omega) = \pm k \delta x = \pm \frac{\omega}{c} \delta x$$

(4.24)

where $\delta x$ is the error in microphone placement position. This gives the potential error in the wavevector

$$\delta K(f) = \frac{2\pi f}{v(f) \pm \delta v(f)} - \frac{2\pi f}{v(f)}.$$  

(4.25)

For example, at a frequency of 2 kHz, using the measured phase data the error in wavevector $\delta K = \pm 0.085$, showing, within experimental error, approximate agreement between the measured and theoretical results.
Another example of the measured band structure, for 3 periods of the same Bragg stack, is shown in Fig. (4.17). The same divergence at higher frequencies is seen as for the 8 period structure discussed above. Now, however, full band gaps cannot be seen to occur in the measured band structure. This implies that the transmission dips seen at these frequencies are only the pre-cursors to band structure, and there are not sufficient periods to open up a full band gap at the BZ edge. As discussed, these results are likely to show a fairly high degree of experimental error, due to the measurement sensitivity of the phase, but they do support the fact that full band gaps open up with increasing number of periods, after first the pre-cursors of band structure are seen in the transmission spectra. This supports the above theoretical discussion of how the transmission spectra vary with number of periods.

4.3.3 Non-equal perturbation and spacing lengths

A comparison of the theoretical and experimental results for structures with non-equal perturbation and spacing lengths is presented here to verify that the model is accurate.
Figure 4.18: The intensity transmission coefficient for an 8 period modulated waveguide structure with a period length of (a) 0.258 m where the perturbations are spaced at twice their length and (b) 0.136 m where the perturbations are spaced by $\ell = 0.05$ m. The ratio of areas in the perturbed and ‘normal’ sections is as in the standard test arrangement. The theoretical predictions (incorporating mass loading) show good agreement with the experimental results.

when the structure is not a simple Bragg stack. In particular, this serves to verify that the mass loading terms introduced above afford correct predictions when the perturbation and spacing lengths are not equal.

Results for two systems, where the length of the perturbations are 0.086 m and the lengths of the spacings are (a) 0.172 m (twice the perturbation length) and (b) 0.50 m are shown in Fig. (4.18). The theoretical predictions (incorporating mass loading) can be seen to show good agreement with the experimental results, indicating that the mass loading terms introduced above are valid for use when the perturbation lengths are not equal to their spacing.

It is interesting to note that the effects of mass loading are much less prevalent when the perturbation spacing is not equal to an integer multiple of the perturbation length (compare Fig. (4.18(b)) with the theoretical predictions excluding mass loading shown in Fig. (4.11)). This can be understood as, in the present case, the mass loading impedance does not occur with a periodicity that is half that of the main structure. Therefore, the very noticeable dip that was seen resulting from this half period effect is no longer present at just below 2 kHz.
Chapter 5

Defect systems

The previous chapter has considered systems with perfect (local) periodicity. If this perfect periodicity is interrupted by the presence of a defect, the transmission properties of the system are altered due to the presence of defect modes. In this chapter, a periodic system with a single defect will be considered where the defect is formed by varying the length of the central element (either perturbation or spacing) of the modulated waveguide structure. Theoretical results will be presented, and verified by experimental investigations.

5.1 Theoretical investigation of defects

In the following section, the mass loading effects introduced in Section 4.3 will be neglected. This is to allow the explanation of the fundamental properties of defect systems based on the main periodic properties of the medium. This analysis would then hold if, for example, the scattering was caused by a change of medium in which the acoustic waves propagate (e.g. a change in gas or liquid type) and not a change in area. It will be subsequently shown that the mass loading effects can be included to obtain agreement with the experimental results.

5.1.1 Occurrence of defect modes

Consider, as an example, the case of a 6 period Bragg stack, of the form considered in Chapter 4. Let the ‘normal’ waveguide sections be denoted $A$ and the perturbed sections $B$. This modulated waveguide structure can then be notated $BABABABABABA$. 
However, as the waveguide is assumed to be terminated at both ends by semi-infinite structures of type $A$, the final $A$ section has no added effect over the termination of the structure, and the notation can be reduced to $BABABABABAB$. Consider doubling the length of the central section giving a structure $BABABAABABAB$ and consider an acoustic wave propagating at the central frequency of the original band gap (that occurs in the Bragg stack with no central defect), $f_c$. If the first half of the structure is considered in isolation, $BABABA$, the back reflected waves are all in phase (see Fig. (4.2(a))) and so sum coherently. The same is true for the waves reflected from the second half of the array if considered in isolation ($ABABAB$). However, the defect introduces an extra half wavelength into the path length for the reflections from the central period. Therefore, the back reflections from the second half of the array are exactly out of phase (at $f_c$) with those from the first half of the array. These therefore interfere destructively, leading to no overall back reflection and hence total transmission at $f_c$.

This effect can also be derived by noting that a single layer of one-half wavelength thickness (such as the central defect layer) has no effect on propagation at $f_c$, and so can be removed from the structure if considering propagation at that frequency. The structure can therefore be reduced as follows

$$BABABAABABAB \rightarrow BABABAABABAB \rightarrow \ldots \rightarrow BABA \rightarrow \mathcal{B} = 1 \quad (5.1)$$

and hence perfect transmission would be expected at $f_c$. At frequencies a little removed from $f_c$, the first and second half of the structure will not be exactly $\pi$ radians out of phase with each other, and so the minimum in transmission associated with a band gap would be still be expected to form. Thus, the transmission coefficient would be expected to exhibit a band gap with a perfect transmission peak at its centre. The defect can be seen as creating a localised spatially confined state that allows acoustic waves with a frequency equal to the central gap frequency to propagate through the structure. This is analogous to the electron defect states found in real crystalline materials.

The transmission is calculated using the transfer matrix method detailed in Chapter 2 and the defect is introduced by modifying the transfer matrix of the central section to have double length as required. The standard test arrangement from Chapter 4 (structure with a period length of 0.172 m formed in a square waveguide with open areas
of 0.054² m² for the unperturbed sections and 0.038² m² for the perturbed sections) will be used as the basis for the structures in this chapter, with the central element of length \( \ell \) being replaced by a defect section of length \( \ell_d \). This arrangement will be termed the standard defect test arrangement with defect length \( \ell_d \). The resulting transmission coefficient from such a structure with \( \ell_d = 2\ell \) is shown in Fig. (5.1). A narrow perfect transmission peak can be seen to occur at the centre of the band gap (i.e. at \( f_c \)) supporting the above discussion. The peak can be described by two factors – its quality factor (Q-factor) and extinction ratio. The Q-factor is a measure of the width of the peak, and is defined as

\[
Q = \frac{f_0}{\Delta f}
\]

where \( f_0 \) is the frequency of the maximum value of the peak, and \( \Delta f \) is the full width at half maximum (fwhm) of the peak. The extinction ratio is normally expressed as the ratio of two optical powers. However, in this case, it is used to describe the ratio of the peak maximum to minimum and is denoted \( \chi \).

In the current case, the Q-factor of the peak is estimated (using a fwhm estimation incorporated in the model) to be \( Q \approx 100 \), which is a high Q-factor quantifying that the peak is indeed very narrow. In the current case, the transmission coefficient does

![Figure 5.1](image_url)

**Figure 5.1:** The intensity transmission coefficient for a 6 period modulated waveguide structure constructed in the standard defect test arrangement incorporating a double length central defect. The theoretical positions of the stop bands for an infinite periodic medium with no defect are also shown (red dotted line). A narrow perfect transmission peak can be seen in the centre of the band gap due to the defect mode.
Defect systems

<table>
<thead>
<tr>
<th>Number of periods</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q-factor, $Q$</td>
<td>10.2</td>
<td>45.3</td>
<td>415</td>
<td>1425</td>
<td>$2.0 \times 10^6$</td>
</tr>
<tr>
<td>Extinction ratio, $\chi$</td>
<td>9.4</td>
<td>85.7</td>
<td>3225</td>
<td>$\sim 10^5$</td>
<td>$\sim 10^{11}$</td>
</tr>
</tbody>
</table>

Table 5.1: The Q-factor and extinction ratio of the defect peak resulting from various periods of modulated waveguide structure constructed in the standard defect test arrangement incorporating a double length central defect. Both the Q-factor and extinction ratio increase with the number of periods to very high values.

not quite reach zero (with the defect peak at the centre of the band gap preventing the transmission at $f_c$ being zero also), but the extinction ratio of the peak is still high, and is estimated from the model to be $\chi = 316$.

5.1.2 Variation with number of periods

It is of interest to consider how the properties of the defect peak vary as the number of periods in the structure is changed. The position of the defect in the band gap is unchanged (as the analysis of defect peak position above holds in general for any number of periods) but the Q-factor and extinction ratio would be expected to vary. The values of these quantities are tabulated for various period numbers in Table (5.1). Note, if an odd number of periods is used, the central defect section (which is doubled in length) is a perturbation rather than a spacing (‘normal’) section of waveguide. However, the above analysis holds and the same form of defect peak is seen.

As the number of periods is increased, both the Q-factor and the extinction ratio increase, reaching very large values for reasonably large numbers ($\gtrsim 10$) of periods. Consider first the Q-factor. As the number of periods is increased in a normal Bragg stack (with no defect), the transmission at the centre of the gap becomes smaller and smaller (tending towards zero for a sufficient number of periods) as at this frequency, the quarter-wave condition is exactly satisfied. The defect is attempting to open up a transmission peak at the centre of this gap and so it is more strongly confined spatially the lower the no-defect Bragg stack transmission is at $f_c$. Stronger spatial confinement leads to narrower peaks, and hence higher Q-factors. Therefore, structures with a larger number of periods lead to narrower peaks and higher Q-factors. Note, the very high Q-factors (for $> 10$ periods) would never be achieved in practice due to small errors in the construction process of the structure. This will be discussed further in Section 5.2.

Now consider the extinction ratio. As the number of periods is increased, the band
gap forms more fully, and so the transmission in the band gap tends to zero over a larger frequency range. As the transmission peak always has a theoretical maximum value of unity (perfect transmission) the extinction ratio is given by

$$\chi = \frac{1}{\alpha_{t_{\text{min}}}}$$  \hspace{1cm} (5.3)

where $\alpha_{t_{\text{min}}}$ is the minimum value of the transmission coefficient in the band gap (beside the defect peak) of the defect system. As the number of periods is increased, this value tends to zero, and so $\chi \to \infty$, as seen in Table (5.1). Thus, defects occurring in Bragg stacks formed with a small numbers of periods are seen to lead to relatively wide defect modes, with low extinction ratios. As the number of periods is increased, the defect modes become narrower (as the defect states become more strongly localised) and their extinction ratio rises.

### 5.1.3 Variation with size of defect

By altering the size of the defect, it is possible to change the characteristics of the defect mode, and hence the transmission peak. These effects are considered in this section. Consider a 6 period modulated waveguide structure constructed in the standard defect test arrangement incorporating various defect sizes, $\ell_d$, with $\ell_d > \ell$ where $\ell$ is the normal (non-defect) perturbation spacing. The transmission coefficients for such systems are shown in Fig. (5.2).

Fig. (5.2) shows that, as $\ell_d$ is increased from unity, the defect peak enters the band gap from the top, and moves from higher to lower frequencies as the defect length is increased, occurring exactly at the centre of the gap for $\ell_d = 2\ell$ as explained above. This behaviour can be explained using the principles discussed above for the double length case. For shorter lengths of defects, the extra half-wavelength phase shift supplied by the defect layer occurs for higher frequency incident waves. When this defect layer introduces an extra half wavelength path, the waves reflected from the second half of the structure will be exactly out of phase with those reflected from the first, and so these reflected waves will again exactly cancel, leading to a perfect transmission peak at the higher frequency that satisfies this condition. As the length of the defect is increased, this frequency falls, causing the perfect transmission peak to move to lower frequencies.
Figure 5.2: The intensity transmission coefficient for a 6 period modulated waveguide structure constructed in the standard defect test arrangement incorporating a central donor-like defect of various lengths ($\ell_d = 1.2\ell, 1.5\ell, 1.8\ell, 2.0\ell, 2.5\ell$). The theoretical positions of the stop bands for an infinite periodic medium with no defect are also shown (black dotted line). A narrow perfect transmission peak is seen in all cases, which enters from the top of the band gap and moves to lower frequencies as the defect length increases.

<table>
<thead>
<tr>
<th>Size of defect, $\ell_d$</th>
<th>1.2$\ell$</th>
<th>1.5$\ell$</th>
<th>1.8$\ell$</th>
<th>2.0$\ell$</th>
<th>2.5$\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q-factor, $Q$</td>
<td>40.8</td>
<td>71.2</td>
<td>87.6</td>
<td>99.6</td>
<td>73.75</td>
</tr>
<tr>
<td>Extinction ratio, $\chi$</td>
<td>4.7</td>
<td>27.2</td>
<td>128.1</td>
<td>316</td>
<td>66.9</td>
</tr>
</tbody>
</table>

Table 5.2: The Q-factor and extinction ratio of the defect peak resulting from a 6 period modulated waveguide structure constructed in the standard defect test arrangement incorporating a central defect of various lengths ($\ell_d = 1.2\ell, 1.5\ell, 1.8\ell, 2.0\ell, 2.5\ell$). Both the Q-factor and extinction ratio increase with proximity of the peak to the centre of the band gap.

as seen. These longer defects are analogous to donor impurities (impurities that donate an electron to the system) in solid state physics.

It is also of interest to consider the properties of the transmission peaks, which are characterised in Table (5.2). Clearly, the Q-factor increases (indicating narrower peaks) as the transmission peak moves closer to the centre of the band gap. At frequencies closer to the centre of the gap, $f_c$, the transmission from the equivalent Bragg stack with no defect is closer to zero than at frequencies away from $f_c$. Therefore, the defect mode is again more confined spatially, and so will have a narrower frequency width. The extinction ratio is also seen to rise with proximity to the centre of the band gap. This is because the transmission peak is then surrounded by regions where the transmission
is closer to zero, leading to higher values of the extinction ratio.

It is also possible to have a defect section that is shorter than the original Bragg stack section, $\ell_d < \ell$. This is equivalent to acceptor impurities in solid state physics, where the defect manifests as a positively charged hole. The transmission coefficients for various lengths of acceptor-type defect in a 6 period modulated waveguide structure constructed in the standard defect test arrangement are shown in Fig. (5.3). The defect peak can be seen to enter from the bottom of the band gap, and as the defect length is reduced, the peak moves to higher frequencies as would be expected, as the extra half-wavelength phase shift supplied by the defect layer is satisfied by higher frequency waves for shorter defect sections. Note, the defect peak appears at the centre of the band gap when the defect length is reduced to zero. This seems to be in contrast with the results of Munday et al. [24]. Although they find the same general trend, with the defect mode (for acceptor-like defects) entering from the bottom of the gap and moving to higher frequencies as the defect length is reduced, they find the defect mode appears in the centre of the gap when the defect length is one-half of the original length, rather than the zero length found here. However, the value found in this work can be justified.

Figure 5.3: The intensity transmission coefficient for a 6 period modulated waveguide structure constructed in the standard defect test arrangement incorporating a central acceptor-like defect of various lengths ($\ell_d = 0.8\ell, 0.5\ell, 0.2\ell, 0$). The theoretical positions of the stop bands for an infinite periodic medium with no defect are also shown (black dotted line). A narrow perfect transmission peak is seen in all cases, which enters from the bottom of the band gap and moves to higher frequencies as the defect length reduces.
Consider the structure denoted as $A$ and $B$ sections as above

$$BABABA_DBABAB$$

(5.4)

where $A_D$ denotes the central defect section. If the length of the defect section tends to zero, Eqn. (5.4) becomes

$$BABABBABAB.$$  

(5.5)

It was seen above that, at the centre frequency of the gap, $f_c$, adjacent pairs of the same structure ($AA$ or $BB$) have no effect and can be cancelled, and so, at this frequency, the structure becomes

$$BABABBABAB \rightarrow BABABBABAB \rightarrow \ldots \rightarrow BABABBABAB \rightarrow B = 1.$$  

(5.6)

Therefore, for a zero length defect, a perfect transmission peak would be expected at the centre of the band gap. Indeed, this system (Eqn. (5.5)) can also be viewed as a 5 period structure with a central (donor-like) defect that is doubled in length, leading to a perfect transmission peak at the centre of the band gap as discussed above. This therefore provides some theoretical justification of this result. An experimental verification will be presented below.

It is also of interest to consider the properties of the acceptor-like defect peaks, whose properties are shown in Table (5.3). Both the Q-factor and extinction ratio are again seen to increase with the peaks proximity to the centre of the gap. However, the values and increases of these quantities are smaller than for the equivalent 6 period donor-like system. This is because the acceptor-like structure is tending to a 5 period system, and so all the band gap features (sharpness, flatness, approximation to zero transmission) are less well defined in this case than for the longer structure. This leads to defect modes
that are less spatially confined and hence wider in frequency as seen.

5.2 Experimental investigation of modelling results

This section presents the results of an experimental investigation into the periodic structures incorporating central defects of the kind discussed above. The experimental techniques used to make these measurements were discussed in Chapter 3. For the reasons set out in Section 4.3.1, mass loading will be included in the theoretical results presented in this section for accurate comparison with the measured data.

5.2.1 Double length central defects

Consider a system constructed in the standard defect test arrangement, with a double length central defect, $\ell_d = 2\ell$. Initially a 4 period structure will be considered and its transmission coefficient is shown in Fig. (5.4) along with theoretical calculations including mass loading.

The general features of the measured results are well modelled by the theoretical calculations (when mass loading is included). Comparison of these results to the theoretical results when mass loading is not included (Section 5.1) show that the mass loading again introduces a large dip at approximately 2 kHz due to the periodicity of the radiation impedance. Some extra oscillations in the transmission coefficient are seen in the measured results that are not predicted by the theory. However, considering the discussion of low period number structures in Section 4.3, these are highly likely to have resulted from reflections from the loudspeaker, and are therefore probably due to experimental error and are not actual features of the system.

It is of interest to analyse the defect peak itself. The Q-factor of the measured defect peak is approximately $Q \approx 18$, which is close to but slightly lower than the theoretically calculated value of 22.8. The measured extinction ratio of $\chi \approx 15$ is also somewhat lower than the theoretical value of 26.3. The measured minimum value of the peak agrees well with the theoretical value. The reduction in extinction ratio is therefore largely due to the reduction in the maximum value of the defect peak (the peak has a maximum transmission coefficient of $\alpha_t \approx 0.7$ compared to the perfect transmission that occurs in theory). This reduction in peak maximum also partially explains the
5 Defect systems

Figure 5.4: The intensity transmission coefficient for a 4 period modulated waveguide structure constructed in the standard defect test arrangement incorporating a double length central defect. Experimentally measured data for such a system is shown along with the theoretical predictions incorporating mass loading. The general agreement between the theory and experimental data is good and a high (although not perfect) transmission defect peak is seen to occur at the centre of the band gap.

decrease in measured Q-factor with respect to the theoretical case. This reduction in peak transmission may be partly due to experimental measurement resolution, but is likely due mainly to small experimental errors in the construction of the modulated waveguide structure. Any small change in the length of a section will lead to small changes in the frequencies that can propagate without loss through different parts of the waveguide structure. Thus, no frequency will, in practice, achieve perfect transmission in the structure. In addition to this, it was seen for the periodic systems that some absorption exists within the waveguide structure. Therefore, even if the accuracy of the structure was such to allow perfect transmission, some absorption would reduce the magnitude of the transmission peak. These experimental variations will also tend to broaden the defect peaks slightly. Despite these experimental imperfections, the Q-factor and extinction ratio are in fairly good agreement with the theoretical predictions.

Consider varying the number of periods in the defect structure. The experimentally measured transmission coefficient for a (a) 6 and (b) 8 period system constructed in
Figure 5.5: The intensity transmission coefficient for a (a) 6 and (b) 8 period modulated waveguide structure constructed in the standard defect test arrangement incorporating a double length central defect. Experimentally measured data for such a system is shown along with the theoretical predictions incorporating mass loading. The general agreement between the theory and experimental data is good with a defect peak seen to occur at the centre of the band gap in both cases. However, for the higher number of periods, this peak is of significantly lower transmission than predicted by theory.

the standard defect test arrangement, with a double length central defect, $\ell_d = 2\ell$ is shown in Fig. (5.5). In both cases, good general agreement is seen between the observed and theoretical results, with defect peaks occurring at the centre of the first band gap. The defect peaks narrow with increasing numbers of periods, following the theoretical results closely. However, the observed peak of transmission also becomes smaller with increased period number, thus significantly decreasing the Q-factor and extinction ratio of the measured defect peaks with respect to their theoretical values (measured Q-factors of 55 and 75 for 6 and 8 periods respectively compared to theoretical values of 81.6 and 391.6 and measured extinction ratios of 59.4 and 188.7 for 6 and 8 periods respectively compared to theoretical values of 189.5 and 2033). These discrepancies are larger than were observed for 4 periods. This is because the spatial confinement of the defect mode is larger for structures with more periods (as discussed) leading to a narrower defect mode, which is more sensitive to small errors in the experimental arrangement, leading to the reduction in peak transmissions seen. Neglecting the peak transmission values, however, excellent agreement of the measured results and the theory is seen.
5.2.2 Variation with size of defect

As the defect modes were seen to be most clearly formed in the above examples when 4 periods of structure were used, this arrangement will be used to illustrate the effects of varying the length of the central defect. Consider first a donor-type defect (where \( \ell_d > \ell \)) in a 4 period modulated waveguide structure constructed in the standard defect test arrangement. The transmission coefficients are shown, for various defect lengths, in Fig. (5.6).

As expected from the above theoretical discussion, the defect peak enters from the top of the band gap, reducing in frequency as the defect length is increased, and passing

![Graphs showing transmission coefficients for different defect lengths](image)

**Figure 5.6:** The intensity transmission coefficient for a 4 period modulated waveguide structure constructed in the standard defect test arrangement incorporating a central donor-type defect of length \( \ell_d = 1.5\ell, 2\ell, 2.5\ell, 3\ell \). Experimentally measured data for such a system is shown along with the theoretical predictions incorporating mass loading. The general agreement between the theory and experimental data is good with a defect peak seen to enter the band gap from the top and move to lower frequencies as the size is increased, occurring at the centre of the band gap for \( \ell_d = 2\ell \).
through the centre of the band gap for $\ell_d = 2\ell$. Good general agreement is again seen between the measured and theoretical results. Note, the measured magnitude of the defect peak falls with proximity to the centre of the band gap. This is because the defect mode is more spatially confined if it occurs closer to the centre of the gap, as discussed, and so small experimental errors in spacing length leading to a reduction in the measured magnitude of the peak have more effect.

Now consider the same system, but with acceptor-type defects ($\ell_d < \ell$). The transmission coefficients are shown in Fig. (5.7). Again, good general agreement is seen between the measured and theoretical results. As the defect is introduced (by reducing

![Figure 5.7: The intensity transmission coefficient for a 4 period modulated waveguide structure constructed in the standard defect test arrangement incorporating a central acceptor-type defect of length $\ell_d = 0.75\ell, 0.5\ell, 0.25\ell, 0\ell$. Experimentally measured data for such a system is shown along with the theoretical predictions incorporating mass loading. The general agreement between the theory and experimental data is good with a defect peak seen to enter the band gap from the bottom and move to higher frequencies as the size is reduced, occurring at the centre of the band gap for $\ell_d = 0$.](image-url)
the length of the defect section from $\ell$) the defect peak can be seen to enter the band gap from the bottom. As the defect length is further reduced, the defect peak rises in frequency, as expected from the above theoretical discussions.

Of importance here is the position of the defect for $\ell_d = 0$. It can be seen from Fig. (5.7) that, for $\ell_d = 0$, the defect peak has moved to the centre of the band gap. This provides experimental evidence for the claim that, for an acceptor-type defect peak to occur at the centre of the band gap, a defect length of zero, and not $0.5\ell$ is required. Indeed, for $\ell_d = 0.5\ell$, the defect peak can be seen (from Fig. (5.7)) to occur well below the centre of the band gap.
Chapter 6

Quasi-periodic systems

This chapter presents the results of a theoretical and experimental investigation into modulated waveguide structures based on quasi-periodic binary sequences so that the waveguide structure has some degree of quasi-periodicity, but is not fully periodic. The transmission properties of these structures will be considered in relation to their periodicities, and other factors influencing the transmission features will be considered. The Fibonacci sequence, Maximum length sequence and Thue-Morse sequence will be considered.

Throughout this chapter, the waveguide structures will be defined in terms of sequences of $A$ (‘normal’) and $B$ (perturbed) layers. The modulated structure will be taken as being formed in a square waveguide with open areas of $0.054^2 \text{ m}^2$ for the unperturbed sections and $0.038^2 \text{ m}^2$ for the perturbed sections as in the standard test structure of Chapter 4. The length of each $A$ or $B$ section will be taken to be $0.086 \text{ m}$. Throughout this chapter, this will be denoted as the standard QP test structure.

6.1 Fibonacci systems

6.1.1 Fibonacci sequences

The Fibonacci numbers were discovered at the start of the thirteenth century by the Italian mathematician Leonardo of Pisa. They obey the simple recursion relation

$$F_n = F_{n-1} + F_{n-2} \quad n > 1$$

(6.1)
with

\[ F_0 = 0 \]
\[ F_1 = 1 \]

forming the sequence

\[ 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 \ldots \] (6.2)

This number sequence has long since fascinated mathematicians due to its occurrence in nature and its many varied properties [48]. For example, the ratio of successive numbers from the sequence approaches the golden mean as \( n \to \infty \)

\[ \frac{F_n}{F_{n-1}} \to \frac{1}{2} \left( 1 + \sqrt{5} \right) \] (6.3)

and the sequence of Fibonacci numbers is periodic if taken modulo any number.

As discussed in Section 1.5 investigations in various branches of physics have been made of structures based on Fibonacci sequences using a recursion relation of the type given in Eqn. (6.1). In this case, if the two structure types (in this work, sections of waveguide with differing cross-sectional area) are denoted \( A \) and \( B \) (where \( B \) denotes a perturbation to the original waveguide) the structure (of order \( n \)) is given by the recursion relation

\[ S_n = \{ S_{n-1}, S_{n-2} \} \] (6.4)

with

\[ S_0 = \{ B \} \]
\[ S_1 = \{ A \} . \]

The sequence of the Fibonacci multi-layer structure definitions up to eighth order is shown in Table (6.1). Clearly, the number of layers follows a Fibonacci sequence (FS) as well, and (from Eqn. (6.1)) is given by \( F_{n+1} \). Thus, only fairly low orders of the sequence would be practical to implement on a scale that would be required to lead to acoustic features in the audible frequency range.
Table 6.1: The sequence of the Fibonacci multi-layer structure definitions up to eighth order. A denotes a ‘normal’ section of waveguide and B a section with a perturbed cross-section (i.e. an expansion or contraction in the waveguide).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Fibonacci layer sequence</th>
<th>No of layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>B</td>
<td>1</td>
</tr>
<tr>
<td>$S_1$</td>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>$S_2$</td>
<td>BA</td>
<td>2</td>
</tr>
<tr>
<td>$S_3$</td>
<td>ABA</td>
<td>3</td>
</tr>
<tr>
<td>$S_4$</td>
<td>BAABA</td>
<td>5</td>
</tr>
<tr>
<td>$S_5$</td>
<td>ABABAABA</td>
<td>8</td>
</tr>
<tr>
<td>$S_6$</td>
<td>BAABAABABAABA</td>
<td>13</td>
</tr>
<tr>
<td>$S_7$</td>
<td>ABABAABABAABABAABABAABA</td>
<td>21</td>
</tr>
<tr>
<td>$S_8$</td>
<td>BAABAABABAABABAABABAABAABAABAABAABAABAABAABAABAABAABAABABA</td>
<td>34</td>
</tr>
</tbody>
</table>

Figure 6.1: The intensity transmission coefficient for a 6th order FS modulated waveguide structure constructed in the standard QP test arrangement. The theoretical positions of the stop bands for an infinite periodic medium with the same individual layer thicknesses are also shown (red dotted line). Clear pseudo-gaps are present in the transmission spectrum, that are spaced either side of the band gaps from an infinitely periodic medium.

6.1.2 Theoretical transmission

Consider first a FS structure of order 6 constructed in the standard QP test arrangement. The sequence of layers is shown in Table (6.1) and the transmission coefficient for this structure is shown in Fig. (6.1). Again, mass loading will, at first, be neglected to allow a discussion of the fundamental transmission features resulting from the FS arrangement. Mass loading will be included subsequently to show agreement with experimental results.
Clear pseudo-gaps can be seen to result. Note, these are termed pseudo-gaps as the medium is not periodic, and so full band gaps (as defined by the Bloch phase) do not form. The resulting gaps can, in fact, be regarded as forming due to localization of acoustic waves in the medium. The gaps occur spaced around the central frequency of the band gap that would result from a Bragg stack with the same layer thicknesses \((ABABABA\ldots\text{with }A\text{ and }B\text{ specified as for the FS structure})\). The transmission of acoustic waves through the structure is now due to the interference of a more complicated sequence of reflections at each area discontinuity in the waveguide.

One way to view the transmission properties is that a wide gap exists in the structure (from around 600 Hz to 1400 Hz), with a series of defect peaks around the centre of the gap. This gap is much wider than that of the equivalent periodic system, but is interrupted by the defect peaks in the centre. The transmission at the centre of this wide gap (equivalent to the centre of the equivalent periodic system gap) can be analysed exactly, using the principles used in the analysis of defects in Chapter 5. Recall that, at \(f_c\), a double layer, \(AA\) or \(BB\), has no effect. The structure (defined in Table (6.1)) can therefore, at \(f_c\), be reduced as follows:

\[
BA\bar{A}\bar{A}\bar{A}\bar{B}\bar{A}\bar{A}\bar{B}A \rightarrow \bar{B}\bar{B}\bar{B}\bar{A}\bar{B}A \rightarrow B\bar{A}A = B.
\]  

(6.5)

Thus, at \(f_c\), the structure acts as if it is composed of only a single \(B\) layer, equivalent to a single period of the Bragg stack considered in Chapter 4. Comparing the transmission for this single period system in Fig. (4.5) with that for the FS system in Fig. (6.1) shows that the transmission at \(f_c\) is indeed the same for the two structure, explaining this peak in the middle of the wider pseudo-band gap. The other defect-like peaks are more difficult to analyse directly, but result due to the exact interference conditions between reflections from each discontinuity in the area of the structure.

In order to elucidate the properties of the structure that give rise to the appearance of the pseudo-gaps, it is useful to consider the periodicity of the structure. This can be conveniently represented using the auto-correlation function (ACF), \(R_{xx}(\tau)\), of the sequence which is a normalised version of the auto-covariance function (ACVF), \(s_{xx}\), normalised by the ACVF value at \(\tau = 0\)

\[
R_{xx}(\tau) = \frac{s_{xx}(\tau)}{s_{xx}(0)}.
\]  

(6.6)
The ACVF is given [49] by time averaging the product of a function, \( x(t) \), and a time-shifted version of itself over a (large) time window, \( T \),

\[
s_{xx}(\tau) = \lim_{T \to \infty} \left[ \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t + \tau) \, dt \right]
\]  

(6.7)

in order to obtain a description of the relation between the function at two times, which, for a stationary ergodic process, is dependent only on the relative time difference and not the absolute times themselves.

The Wiener-Khinchine theorem states [49] that the auto power spectral density (APSD), \( S_{xx} \), of a function is the fourier transform of the ACVF of the function

\[
S_{xx}(\omega) = \int_{-\infty}^{\infty} s_{xx}(\tau) e^{-j\omega \tau} \, d\tau
\]

where the ACVF is given by Eqn. (6.7). Substitution of the ACVF function into Eqn. (6.8) gives the APSD

\[
S_{xx}(\omega) = \lim_{T \to \infty} \left[ \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t + \tau) \, dt \right] e^{-j\omega \tau} d\tau
\]

(6.8)

where \( F \) denotes the fourier transform of the function contained within its argument.

By the convolution theorem,

\[
\mathcal{F}\left\{ \int g(t)h(t + \tau) \, dt \right\} = G(\omega)H^*(\omega)
\]

(6.10)

where \( G(\omega) \) and \( H(\omega) \) are the fourier transform pairs of \( g(t) \) and \( h(t) \) respectively, and substitution of this into Eqn. (6.9) gives the APSD

\[
S_{xx}(\omega) = \lim_{T \to \infty} \left[ \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\omega t} \, dt \right]^2
\]

(6.11)

A ‘raw estimate’ of the APSD, which is equal to the full expression for periodic functions,

\footnote{For a periodic signal, the integral should be performed over an integer number of periods of the signal.}
is given by

\[ S_{xx}(\omega) \approx \frac{1}{T} \left| \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\omega t} \, dt \right|^2 \]  

(6.12)

for a time interval, \( T \), that is large (or equal to an integer number of periods for a periodic function). Thus

\[ S_{xx}(\omega) = \frac{X(\omega)X^*(\omega)}{T} \]  

(6.13)

where \( X(\omega) \) is the fourier transform of \( x(t) \) and \( * \) denotes the complex conjugate. From Eqn. (6.8), the ACVF of the function \( x(t) \) can therefore be calculated

\[ s_{xx}(\tau) = \mathcal{F}^{-1}\left( \frac{X(\omega)X^*(\omega)}{T} \right) \]  

(6.14)

where \( \mathcal{F}^{-1} \) denotes the inverse fourier transform. The ACF is then calculated from Eqn. (6.6).

In the current case, the sequence defining the FS structure can be represented in the form of a digital signal, with \( A \) sections denoted ‘0’ and \( B \) sections denoted ‘1’. The ACF then characterises the periodicity of the perturbations where \( \tau \) corresponds to a single spacing length. For example, for the periodic systems considered in Chapter 4, the ACF exhibits unit peaks at every even value of \( \tau \), showing a periodicity of 2 layers.

The ACF of the 6th order FS is shown in Fig. (6.2). Strong periodicities can be

![Figure 6.2: The auto-correlation function of a 6th order Fibonacci sequence.](image-url)
seen at spacings of 3 and 5, with some periodicity also visible at 2 spacings. It is likely that the periodicities of 2 and 3 help give rise to the wide pseudo-gaps seen, whilst the double $A$ layers (which appear in the ACF as periodicities of 3 spacings) give rise to the central defect peaks interrupting these gaps as discussed above. The ACF of this system will be contrasted to other order FS structures and also MLS and Thue-Morse structures below. It should be noted that, although the ACF is a useful tool in examining the periodicity of the structure, there is no obvious relationship between the ACF and the transmission coefficient of the structure (for example one based on Fourier transform techniques) as the sound wave interacts with scattering elements one after another, rather than simultaneously.

Now consider the transmission coefficient and ACF for a $7^{th}$ order FS structure, shown in Fig. (6.3). The $7^{th}$ order structure sequence is shown in Table (6.1). Pseudo-band gaps are still seen to form, which are well defined compared to the $6^{th}$ order structure, with flatter stop bands and steeper roll-offs. This is not surprising as the $7^{th}$ order structure is significantly longer than the $6^{th}$ order structure, and so any features resulting from periodicity in the medium would be expected to be better developed, as seen here.

![Figure 6.3](image-url)

**Figure 6.3:** (a) The intensity transmission coefficient for a $7^{th}$ order FS modulated waveguide structure constructed in the standard QP test arrangement. The theoretical positions of the stop bands for an infinite periodic medium with the same individual layer thicknesses are also shown (red dotted line). Clear pseudo-gaps are present in the transmission spectrum, with more defined features than for the $6^{th}$ order system (Fig. (6.1)), and a perfect transmission peak is seen at the original band gap centre frequency. The ACF of the $7^{th}$ order FS is shown in (b).
At $f_c$ (the centre frequency of the original band gap), a perfect transmission peak is now seen. Performing the same analysis as above, at $f_c$, the structure reduces as follows:

$$ ABABABABABABABABA \rightarrow ABABABABABA $$

$$ \rightarrow ABABAAB \rightarrow ABB \equiv 1 $$ \hspace{1cm} (6.15)

explaining the perfect transmission peak at $f_c$. The other central peaks are due to reflections at other frequencies which are not completely in phase throughout the structure leading to lower transmission defect peaks. The ACF indicates the presence of periodicities of the structure at 2, 3 and 5 spacings as before, and also at 8 spacings (note, these periodicities are following a Fibonacci sequence). Thus, these periodicities would seem to be important for the creation of the pseudo-gaps seen in this case.

It is of interest to consider the exact structure of the central defect peak. This is shown, for various orders of FS, in Fig. (6.4). Note how the central section of each bottom plot contains a scaled (reduced in width) version of the central section of the plot directly above it. The bottom plots are for FS structures of 3 orders higher than the equivalent top plots. This phenomena is termed the scaling relation, and it has

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figures/figure6.4.png}
\caption{The transmission coefficient around the central defect peak for FS structures constructed in the standard QP test arrangement with orders 4 to 9. Comparison of the plots displays the scaling behaviour of the Fibonacci structures transmission coefficients.}
\end{figure}
been used [40] as evidence of localization of waves in Fibonacci crystals, explaining the existence of the pseudo-band gaps seen. It is encouraging, therefore, that these features are observed in the current model.

6.1.3 Experimental results

This section presents the results of an experimental investigation into modulated waveguide structures based on Fibonacci sequences. The transmission coefficients measured from the experimental data and theoretical calculations (incorporating mass loading effects) for Fibonacci sequence structures of orders 4 to 7 constructed in the standard QP test arrangement are shown in Fig. (6.5). Good general agreement is seen between the experimental and theoretical (including mass loading) results. For low orders of structure sequence, the measured transmission coefficient can be seen to display extra oscillations

![Graphs showing transmission coefficients for Fibonacci orders 4 to 7.](image-url)

**Figure 6.5:** The intensity transmission coefficient for 4th, 5th, 6th and 7th order FS modulated waveguide structures constructed in the standard QP test arrangement. Good general agreement (especially for high orders) is seen between the measured and theoretical results.
that are not present in the theoretical results. These can again be classed as due to reflections from the loudspeaker (see the discussion in Chapter 4) and are therefore just a result of experimental error.

For the higher order structures, the higher values of transmission can be seen to be reduced somewhat below their theoretical values. This is most likely due to absorption within the waveguide structure, which is more important for the higher order structures as the propagation distance is much longer (the structures are much bigger). The central defect transmission peak of the 7th order FS system is substantially reduced below the theoretical value. This can be understood as the same mechanism which reduces the defect peak magnitude discussed in Chapter 5. The perfect transmission holds only for acoustic waves propagating through the structure at $f_c$ leading to a narrow defect peak. However, if there is some error in the spacings of the blocks, there is some variation in the frequencies that couple to this defect mode, thus reducing the magnitude of the transmission at $f_c$ and reducing the Q-factor and extinction ratio of the defect peak with respect to the theoretical value.

Otherwise the agreement between experiment and theory is very good, especially for the higher order structures, where the features are more clearly defined. In particular, the triple-peak structure of the defect peaks in the centre of the band gap for the 6th and 7th order systems is very clearly defined, and the scaling behaviour can be observed between the 4th and 7th order structures about the central defect peak. The pseudo-band gaps are also very clearly defined in the 7th order system. The mass loading (in addition to the small frequency shift of the main features) can again be seen to result in a significant dip (when many perturbation sections are used) at just below 2 kHz. The quality of the fit to this transmission dip gives further justification that the empirical solution of the radiation impedance performed in Chapter 4 is valid for use in this work. An important influence of the mass loading is that the main initial band gap width is slightly reduced (especially above the central defects). This results in a performance with mass loading that is slightly worse than when mass loading is not included (assuming the aim is to achieve wide band gaps). This effect is, however, small.

This, to the best of the authors knowledge, presents the first experimental observation of the acoustical transmission properties of macroscopic modulated systems based on Fibonacci sequences, and validates the theoretical predictions. Furthermore, the ex-
Experimental arrangement used for the construction of the FS system is accurate enough to display the detailed peaks and dips associated with this system, suggesting that the system may be suitable for practical use if these defined features are required. Methods for combining FS systems with other systems to improve their characteristics will be considered further in Chapter 7.

6.2 MLS systems

6.2.1 MLS sequences

The Fibonacci layer sequences listed in Table (6.1) represent a binary number sequence. It is of interest to also consider some further binary number sequences that could provide a potential basis for determining multi-layer structure definitions. A possible example is a Maximum length sequence (MLS), a binary pseudo-random sequence of length \( L = 2^n - 1 \) where \( n \) is an integer, which can be efficiently generated using shift registers. A MLS is spectrally flat except at dc and has an autocorrelation that approaches zero at all but zero delay as the order of the sequence increases. Thus, the sequence approximates to white noise, and is an example of a pseudo-random sequence.

A MLS sequence can be used to determine the structure of the waveguide by assigning a ‘B-layer’ (perturbation) whenever the MLS takes unit value, and an ‘A-layer’ when the sequence takes zero value. The sequence of the multi-layer structure definitions provided using MLS sequences up to seventh order is shown in Table (6.2). Again, due to the large numbers of layers associated with these sequences, practical structures would have to be limited to low order sequences.

6.2.2 Theoretical transmission

Consider the 3rd, 4th and 5th order MLS structures constructed in the standard QP test arrangement. The transmission coefficients and ACF of the structure sequences are shown in Fig. (6.6).

From Table (6.2), the 3rd order structure is formed from the layers \( BAABABB \). At the centre frequency of the original band gap, \( f_c \), the structure reduces (using the same methodology as above) to

\[
BAABABB \rightarrow BAABA \equiv 1
\]
Table 6.2: The sequence of the MLS multi-layer structure definitions up to seventh order. $A$ denotes a ‘normal’ section of waveguide and $B$ a section with a perturbed cross-section (i.e. an expansion or contraction in the waveguide).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>MLS layer sequence</th>
<th>No of layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_2$</td>
<td>BAB</td>
<td>3</td>
</tr>
<tr>
<td>$S_3$</td>
<td>BAABABB</td>
<td>7</td>
</tr>
<tr>
<td>$S_4$</td>
<td>BAAABABBABABBABB</td>
<td>15</td>
</tr>
<tr>
<td>$S_5$</td>
<td>BAAAAABAAABBABBBBAABBBABABABA</td>
<td>31</td>
</tr>
<tr>
<td>$S_6$</td>
<td>BAAAAABAAAABBBBBBBBBBBBBBBBABBAAABB</td>
<td>63</td>
</tr>
<tr>
<td>$S_7$</td>
<td>BAAAAAABAAAABBBBBBBBBBBBBBBBBBBB</td>
<td>127</td>
</tr>
</tbody>
</table>

accounting for the perfect transmission peak at $f_c$. Pseudo-gaps are seen either side of this central peak. However, this is mainly due to the small number of elements in the structure, meaning it is easier to develop small transmission minima as seen.

For the higher order structures, significantly more peaks and dips are present in the transmission spectra, and no discernable pseudo-gaps exist over any significant frequency range. This is most likely due to the random nature of the sequence. Consider the ACF of the three sequences (shown in Fig. (6.6)). The ACF takes unit value at zero delay. For all other delays, however, it takes the value $-1/N$ where $N$ is the length of the sequence, showing that, for large $N$, the MLS sequence approximates to a random sequence, with no defined periodicities.

For the 5th order structure, it can again be shown that the structure has no effect at $f_c$. For other frequencies both in this and the 4th order structure, the lack of any significant periodicity in the layer sequence leads to a complex series of phase relations between the various reflections, leading to the complex series of peaks and dips seen, meaning no discernable pseudo-band gaps form. The transmission spectra is more complicated, with more peaks and dips, for the higher order structures due to a greater degree of randomness in the MLS sequence used to generate the structure. Note, the transmission coefficient is not in any way random – in fact, symmetries can still be seen. However, the lack of periodicity leads to a very complex sequence of reflections yielding many peaks in dips of varying magnitudes in the transmission as seen.

These results show that some periodicity is important in the formation of band gaps.
Figure 6.6: The intensity transmission coefficient (left) for a 3rd, 4th and 5th order MLS modulated waveguide structure constructed in the standard QP test arrangement. The theoretical positions of the stop bands for an infinite periodic medium with the same individual layer thicknesses are also shown (red dotted line). Pseudo-gaps in the transmission spectrum are present for the third order structure only, but not for any of the higher order structures. The ACF is also shown (right) for the MLS sequences that defined the structure layer sequence.
and pseudo-band gaps. The degree of periodicity required will be considered further by investigation of structures based on Thue-Morse sequences in Section 6.3.

### 6.2.3 Experimental results

To verify the above discussions, experimental measurements were again made. The 4th order MLS structures was constructed, and the measured transmission coefficient is shown in Fig. (6.7). Good agreement is seen between the measured and theoretical results, providing verification for the above theoretical discussions.

The narrow peaks are well modelled by the predictions of the theory, although two peaks (at about 500 Hz and 700 Hz) can be seen to actually have higher transmissions than predicted. This is unusual, as it has been seen above that residual absorption mechanisms within the waveguide lead, in general, to a small reduction in measured transmission with respect to the theoretical values. The extra transmission seen here is most likely due to small experimental errors. One possible source of such error is the creation of double-length perturbation blocks (BB layers). In the waveguide construc-

![Figure 6.7](image_url): The measured intensity transmission coefficient for a 4th order MLS modulated waveguide structure constructed in the standard QP test arrangement. The theoretical (including mass loading) transmission coefficient is also shown. Good agreement is seen between the measured and theoretical results.
tion, these were formed by placing two single perturbation blocks together. There may therefore be small reflections at the join of the two blocks, which should not occur in theory. Also, any small gaps at the join may give rise to resonant effects, and small errors in the specified length of the block. These effects may combine to give rise to the discrepancies seen.

6.3 Thue-Morse systems

6.3.1 Thue-Morse Sequences

Another binary sequence is the Thue-Morse sequence with applications in number theory, combinatorics and differential geometry. It is defined [50] by the recursion relation

\[ S_n = \{ S_{n-1}, S_{n-1}^+ \} \quad n \geq 1 \] (6.17)

where \( S_n^+ \) denotes a second recursion relation

\[ S_n^+ = \{ S_{n-1}^+, S_{n-1} \} \] (6.18)

with

\[
\begin{align*}
S_0 &= 0 \\
S_0^+ &= 1.
\end{align*}
\]

This recursion relation can be implemented by the substitution map

\[
\begin{align*}
0 &\to 01 \\
1 &\to 10
\end{align*}
\] (6.19) (6.20)

giving the sequence

\[ 0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \ldots \] (6.21)
Associating the ‘A-layers’ with 0 and the ‘B-layers’ with 1 as before allows this sequence to also be used to determine the structure of the waveguide. The sequence of the Thue-Morse multi-layer structure definitions up to seventh order are shown in Table (6.3).

Here, the number of layers can be seen to be given by \(2^n\) where \(n\) is the order of the sequence, thus again limiting practically achievable structures to low orders. The interesting property of the Thue-Morse layer sequence is that it is believed [50] to be more aperiodic than the Fibonacci layer sequence (its Fourier spectrum is more continuous). In terms of its quasi-periodicity, it would therefore be expected to lie somewhat between the quasi-periodic Fibonacci structure and the pseudo-random MLS structure. This may therefore offer some insights into the mechanisms at work in the creation of pseudo-band gaps for these structures.

### 6.3.2 Theoretical transmission

Consider the 3\textsuperscript{rd}, 4\textsuperscript{th} and 5\textsuperscript{th} order Thue-Morse structures constructed in the standard QP test arrangement. The transmission coefficients and ACF of the structure sequences are shown in Fig. (6.8). Consider the ACFs of the various order sequences. Clear peaks are seen, indicating a degree of periodicity in the structure, with the dominant components appearing at 3, 6, 9, \ldots spacings. However, the ACF has significant contribution at other spacings, suggesting a degree of randomness in the system above that of the

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Thue-Morse layer sequence</th>
<th>No of layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_0)</td>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>(S_1)</td>
<td>AB</td>
<td>2</td>
</tr>
<tr>
<td>(S_2)</td>
<td>ABBA</td>
<td>4</td>
</tr>
<tr>
<td>(S_3)</td>
<td>ABBABAAB</td>
<td>8</td>
</tr>
<tr>
<td>(S_4)</td>
<td>ABBABAABBAABABBA</td>
<td>16</td>
</tr>
<tr>
<td>(S_5)</td>
<td>ABBABAABBAABBBABABABABAABAB</td>
<td>32</td>
</tr>
<tr>
<td>(S_6)</td>
<td>BABABABABAABABABABABABAABBA</td>
<td>64</td>
</tr>
<tr>
<td>(S_7)</td>
<td>BABABABAAABABBAABABABABABABABBA</td>
<td>128</td>
</tr>
</tbody>
</table>

Table 6.3: The sequence of the Thue-Morse multi-layer structure definitions up to seventh order. A denotes a ‘normal’ section of waveguide and B a section with a perturbed cross-section (i.e. an expansion or contraction in the waveguide).
Figure 6.8: The intensity transmission coefficient (left) for a $3^{\text{rd}}$, $4^{\text{th}}$ and $5^{\text{th}}$ order Thue-Morse modulated waveguide structure constructed in the standard QP test arrangement. The theoretical positions of the stop bands for an infinite periodic medium with the same individual layer thicknesses are also shown (red dotted line). Pseudo-gaps are seen, separated by significant defect peaks. The ACF is also shown (right) for the Thue-Morse sequences that defined the layer structure.
Fibonacci systems. This supports the viewpoint above that the Thue-Morse system has a degree of aperiodicity somewhat between the quasi-periodic Fibonacci structure and the pseudo-random MLS structure.

It would be expected that this is manifested in the transmission characteristics of the structure, and indeed this is seen in Fig. (6.8). Pseudo-gaps are seen to form, but these are separated by significant amounts of defect peaks. Due to the properties of the sequence, sets of double layers are adjacent which will always cancel at $f_c$ (see Table (6.3)) leading to a central defect-type peak with perfect transmission. However, other peaks also exist, and the number of these peaks increases with increasing order of structure (as there are more possible interference paths that give rise to high transmission through the structure due to destructive interference of the back reflected waves). Again, for the higher order structures, the features of the transmission coefficient are more clearly defined, with flatter pseudo-gaps with steeper roll-offs, as would be expected due to the increased length of the structure.

These results, coupled with those for the Fibonacci and MLS structures, suggest that the quasi-periodicity of the structure is indeed important in obtaining pseudo-band gaps and the degree of periodicity is important in determining the width of these pseudo-gaps and the number and size of defect peaks interrupting the gaps. The widest most well-formed gaps are observed for the Fibonacci structure, while the MLS structure shows the least examples of gap formation. The Thue-Morse systems are somewhat between the two, as would be expected for a system whose degree of aperiodicity lies between that of the Fibonacci and MLS sequences.

### 6.3.3 Experimental results

This system was also investigated experimentally, and the measured transmission coefficient for the 4th order system is shown in Fig. (6.9). Again, good general agreement is seen between the features of the experimentally measured and theoretical transmission coefficients. Slight discrepancies are seen at higher frequencies (the uppermost pseudo-gap in the range of the graph is seen to be slightly displaced in frequency from the theoretical position) but this effect is small, and is most likely due to experimental error. Again, several perturbation blocks were placed together to achieve $BB$ sections, and this may have contributed to these errors.
Figure 6.9: The measured intensity transmission coefficient for a $4^{\text{th}}$ order Thue-Morse modulated waveguide structure constructed in the standard QP test arrangement. The theoretical (including mass loading) transmission coefficient is also shown. Good general agreement is seen between the measured and theoretical results, particularly at lower frequencies.
Chapter 7

Heterostructures

The previous chapters have considered the acoustic properties of several types of area modulated structures in isolation. In this chapter, the possibility of combining these structures will be investigated.

In solid state physics, a heterostructure is defined [51] as the structure resulting when layers of semiconductors with different properties are deposited on top of each other. In this work, the term will be used analogously to refer to a composite structure resulting when two or more modulated waveguide structures are combined, and this chapter presents a theoretical investigation into some properties of these systems. The individual structures can be combined in series (one after another) and these systems will be considered incorporating both periodic and Fibonacci structures. The individual structures can also be combined in parallel (within each other) leading to fractal type systems, which will also be considered. Finally, a method to optimise the transmission properties of an array using a genetic algorithm will be considered. Although not strictly a heterostructure, it is included in this chapter as an alternative to the other methods introduced. In this chapter the standard test arrangement refers to a structure formed in a square waveguide with open areas of $0.054^2 \text{ m}^2$ for the unperturbed sections and $0.038^2 \text{ m}^2$ for the perturbed sections.

7.1 Periodic heterostructures

Consider forming a heterostructure with the aim of increasing the width of the band gap over that which can be obtained using a single Bragg stack. Consider initially a
heterostructure formed from 2 periodic Bragg stacks, one with a band gap centered at 1 kHz, and the other with a band gap centered at 1.5 kHz. The transmission coefficients for these single Bragg stacks are shown in Fig. (7.1(a)). The band gaps of these structures just overlap. It would therefore be expected that if these two structures were placed in series to form a periodic heterostructure a band gap would exist over the entire range covered by the first band gap of each individual structure. This is seen to occur in practice in Fig. (7.1(b)). The transmission outside of this widened band gap reveals a complicated pattern of peaks and dips, much more complicated than for the individual Bragg stacks. This is due to the increased number of reflections (due to the extra length of the structure) but also to the fact the reflections throughout the second half of the array will have a more complicated phase relation with those through the first half of the array (due to the difference in layer thickness). The total transmission is therefore not just the simple product of the transmissions due to the component structures – the second structure has an influence on the first and vice versa.

The wide band gap, however, is still seen to form. This seems to be an effective way to generate wide band gap structures, and by introducing more layers, it is possible to increase the band width further. As an example, a further 8 period Bragg stack with $f_c = 2.2$ kHz is added to the heterostructure and the resulting transmission is shown in

![Graph](image)

**Figure 7.1:** (a) The intensity transmission coefficients for an 8 period Bragg stack constructed in the standard test arrangement with band gap centre frequencies of $f_c = 1$ kHz and $f_c = 1.5$ kHz and (b) the intensity transmission coefficient for a heterostructure formed from the combination of these two systems in series. A large increase in the width of the band gap is seen for the heterostructure.
Fig. (7.2). The band gap width is seen to be significantly increased over the 2 element heterostructure shown in Fig. (7.1(b)).

Further Bragg stacks could be concatenated into the heterostructure to extend the band gap in either direction, and this process could, in theory, be used to obtain arbitrarily wide band gaps. It must be noted, however, that this increase in band width requires a significant increase in total length of the structure. In the present example, the structure with $f_c = 1$ kHz has a total length of 1.37 m. Incorporating the structure with $f_c = 1.5$ kHz increases the total heterostructure length to approximately 2.3 m, and incorporating the structure with $f_c = 2.2$ kHz increases this length further to approximately 2.9 m. Increases in band width to lower frequencies are even more costly, as the period length increases with decreasing centre frequency, and the band gaps are narrower at lower frequencies (see the discussion in Chapter 4). The practical value of this procedure may therefore be somewhat limited, especially at low frequencies.

### 7.2 Fibonacci heterostructures

It is also possible to construct heterostructures incorporating Fibonacci structures as one or more constituents of the heterostructure. This technique has previously been used [37]
in optical photonic crystals to obtain broad reflectivity bands. Consider combining a Fibonacci structure with a periodic one (Bragg stack) where both structures are formed in the standard test arrangement with $f_c = 1$ kHz. The structures are notated $(S_n)(S_2)^m$ where $n$ is the order of the Fibonacci structure and $m$ is the number of periods of the Bragg stack (a second order Fibonacci structure, $S_2$, being identical to a Bragg stack arrangement).

As an example, the transmission from a structure $(S_6)(S_2)^5$ is shown in Fig. (7.3), along with the transmissions of the individual structures. Consider the transmissions of the individual structures (Fig. (7.3(a))). The band gaps of the two structures overlap sufficiently so that the Bragg stack can largely compensate for the central defect peaks seen with the Fibonacci structure. However, the heterostructure transmission shows small residual peaks resulting from FS central defect peaks and also the band edge peaks from the Bragg stack. These peaks result as the transmission is not close enough to zero for either structure in its band gap region (the order of the FS and the number of periods of the Bragg stack is too small). Note, these peaks (particularly the central ones) are fairly small in magnitude (the central peak has an intensity transmission coefficient of $\alpha_t \approx 0.01$ corresponding to a reduction in transmitted level of 20 dB) and if these can be tolerated, this heterostructure provides an effective way to widen the band gap.

![Figure 7.3](image)

**Figure 7.3:** (a) The intensity transmission coefficients for a $6^{th}$ order Fibonacci structure and a 5 period Bragg stack both constructed in the standard test arrangement with band gap centre frequencies of $f_c = 1$ kHz and (b) the intensity transmission coefficient for a heterostructure formed from the combination of these two systems in series. An increase in the width of the band gap is seen for the heterostructure, but with some residual peaks remaining in the gap.
with relatively small total structure thicknesses. For example, this structure has a total thickness of less than 2 m, which is smaller than the required thicknesses of the periodic heterostructure systems considered above.

The observed peaks in the heterostructure band gap can be reduced significantly by using higher order structures. An example of the structure \((S_7)(S_2)^7\) is shown in Fig. (7.4). The stop bands for the individual structures can be seen (Fig. (7.4(a))) to have a larger degree of overlap than for the previous heterostructure considered, and also have lower transmission over a larger frequency range. Correspondingly, the transmission of the heterostructure exhibits a wide stop band, which is much flatter than for the heterostructure shown in Fig. (7.3(b)), suggesting that heterostructures incorporating Fibonacci sequences can be used to obtain larger transmission stop bands. However, it should be noted that this structure requires a length of approximately 2.9 m, whereas a stop band covering a similar frequency range could be obtained with a periodic heterostructure of length 2.5 m. If, however, some small transmission peaks can be tolerated within the stop band, low-order Fibonacci heterostructures are able to offer wider stop bands than can be obtained using similar lengths of periodic structures.

An infinite number of further examples can be created, based on the exact desired transmission characteristics of the heterostructure and the available thickness. As a fur-
ther example, the system proposed for optical photonic crystals by Dong et al. [37] will be investigated. This example couples two Fibonacci systems with a periodic system where multiple periods of low (3rd and 4th) order Fibonacci systems are used. Using the notation as above, the complete heterostructure can be defined as \((S_3)^4(S_4)^2(S_2)^7\). The individual transmissions for these structures and that for the composite heterostructure are shown in Fig. (7.5). A wide flat stop band is seen for the heterostructure, in agreement with the optical results of Dong et al. [37]. Note, outside the range of the band gap of the periodic system (the Bragg stack) neither Fibonacci system has transmission approaching zero. This is, however, a good example of how the reduction in transmission of both Fibonacci systems complement each other to provide a wide stop band for the heterostructure, even with low order constituent Fibonacci systems.

The agreement between these results and those of the equivalent optical system helps to illustrate the analogy between the fields of photonic (optical) and phononic (acoustic) crystals. This analogy holds as the underlying phenomena are due to the interference of waves, and these findings help to support the view that structures investigated in one field can have potential use in the other. Indeed, Dong et al. [37] showed that the structure considered here should have a wide omni-directional band resulting in high reflectivity over a large angular range for both s- and p-polarizations. Although the

![Figure 7.5](image_url)

**Figure 7.5:** (a) The intensity transmission coefficients for the Fibonacci structures \((S_3)^4\) and \((S_4)^2\) and also a 7 period Bragg stack, all constructed in the standard test arrangement with band gap centre frequencies of \(f_c = 1\ \text{kHz}\) and (b) the intensity transmission coefficient for a heterostructure formed from the combination of these three systems in series. A wide flat stop band is seen for the heterostructure.
polarization argument is not applicable in airborne propagation of acoustic waves (as they are longitudinal), this system would therefore be expected to provide a wide stop band over a range of angles of incidence of the sound, which is a desirable property for practical applications of such structures.

7.3 Fractal systems

7.3.1 Single fractal systems

The above section has demonstrated how the band gap of the modulated waveguide structure can be extended by placing several structures (with band gaps occurring at complementary frequencies) in series. The limitations of this method were mainly judged to be the required length of the structures. It would therefore be advantageous if additional structures could be created in parallel with (i.e. created within) the original structure. To investigate this, a fractal type system is modelled, where extra perturbations to the area are introduced along the main structure, as represented schematically in Fig. (7.6). If desired, the fractal analogy can be extended by adding extra small perturbations at the centre of the newly created perturbations, giving a higher order fractal system.

Consider initially a 10 period Bragg stack with a central band frequency of $f_c = 1$ kHz constructed in the standard test arrangement modified into a fractal system where the fractal perturbations have a change in area $\delta S = S_1/4$ and a thickness $\delta a = 2a/3$ so that the fractal perturbations are one third as thick as each total layer thickness. The transmission coefficient for this system (with and without the fractal perturbations) and the associated band structure is shown in Fig. (7.7). The addition of the extra fractal structure can be seen to open up a band gap around 2 kHz. It also slightly suppresses

![Figure 7.6: A schematic representation of the variation of cross-sectional areas in a fractal-type system.](image-url)
the original band gaps (at 1 kHz and 3 kHz) reducing their width.

The new band gap occurs at 2 kHz as the fractal perturbations have introduced a new periodicity into the structure that is half that of the original periodicity of the Bragg stack, leading to a band gap at twice the frequency of the lowest band gap of the Bragg stack \(2f_c\), as seen. As would be expected, decreasing the variation of area of the fractal perturbations, \(\delta S\), reduces the width of the extra band gap. In addition, the original band gaps are less suppressed in width if a smaller value of \(\delta S\) is used. This is because the impedance discontinuity between each section and the corresponding fractal perturbation is smaller, and hence it has less effect.

In order to widen the original band gap, it is desired to move the new band gap down in frequency such that it acts with the original band gap to widen this original gap. It may naively be thought that this could be achieved by increasing the length of each fractal perturbation centred in each layer, so that it satisfied the condition of \(\lambda/4\) for lower frequencies. However, the periodicity of such a fractal system, with arbitrary length within each section, will still have a period length equal to \(2a\), and so the fundamental band gap due to this periodicity will still result at \(2f_c\). In order to achieve a variation in this band gap position, it is necessary to impose a periodicity of fractal-type perturbations that is incommensurate with the periodicity of the original structure.
Imagine dividing each layer of the original structure into 3 sections, and placing a fractal perturbation once every 4 sections, as shown schematically in Fig. (7.8(a)). This imposes a second period on the structure which is twice that of the original period. The transmission coefficient and associated band structure of the infinitely periodic system are shown in Figs. (7.8(b) and (c)).

Considering the band structure (Fig. (7.8(c))), the fractal perturbations can be seen to open up several new band gaps (at around 500 Hz, 1500 Hz, 2000 Hz, 2500 Hz etc.)

**Figure 7.8:** (a) A schematic representation of the variation of area in a fractal type system where the period of the fractal perturbations is incommensurate with the period of the original structure. The fractal perturbations impose a second period, \( \Lambda' \), on the structure which is twice that of the original period, \( \Lambda \). The resulting (b) intensity transmission coefficient and (c) associated band structure is shown for a 10 period Bragg stack with a central band frequency of \( f_c = 1 \) kHz constructed in the standard test arrangement modified with the fractal perturbations as shown in (a) where the fractal perturbations take the parameters \( \delta S = S_1/4 \) and \( \delta a = 2a/3 \). The fractal perturbations open up several new band gaps (at around 500 Hz, 1500 Hz, 2000 Hz, 2500 Hz etc.) for an infinitely periodic structure, although a significant dip in transmission is only seen at 1500 Hz due to these new band gaps.
for the infinitely periodic structure, where a single period is defined by $\Lambda'$ in Fig. (7.8(a)). The gaps at 500 Hz, 1500 Hz, 2500 Hz result directly from the doubled period length imposed (leading to the first band gap at 500 Hz and the higher order band gaps at odd-integer multiples of this frequency). Note, the gap at 500 Hz is very small (it can barely be resolved in the band structure) and so the resulting dip in transmission is also very small. The frequencies are shifted slightly from the exact values expected due to the interplay between the two periodicities of the system. This is also responsible for opening up the unexpected band gap at 2000 Hz, which would not result due to either periodicity in isolation.

These new band gaps are very narrow except at 1500 Hz. Correspondingly, a significant dip in transmission with respect to that of the original structure is only seen at 1500 Hz. The original band gap at 1 kHz also shows some suppression in width compared to that of the original structure. This arrangement moves the significant dip provided by the fractal perturbations closer to the fundamental band gap of the non-fractal system than was achieved in Fig. (7.7) as desired. However, to achieve a wider single band gap these two stop bands would need to be overlapped in frequency. Considering Fig. (7.8(b)), this may be achieved if the centre of the stop band resulting from the fractal perturbations was reduced to approximately 1200 Hz.

Again, imagine dividing each layer of the original structure into 3 sections, and placing a fractal perturbation once every 5 sections, as shown schematically in Fig. (7.9(a)). The transmission coefficient and associated band structure of the infinitely periodic system are shown in Figs. (7.9(b) and (c)). This gives a new period repeat distance for the placing of the fractal perturbations of 2.5 times the original periodicity, which, if the original structure has a fundamental band gap frequency of 1 kHz, should give new band gaps at 400 Hz and hence also at 1200 Hz as desired.

The interplay of the two periodicities can be seen to cause a much more complex band structure than was previously seen. The main stop band in transmission due to the fractal perturbations is seen to move towards the fundamental band gap of the Bragg stack. However, its width is somewhat suppressed and a narrow transmission band is seen to result between the two stop bands. Considering the band structure, the original Bragg stack band gap is seen to occur at the centre of the Brillouin zone (BZ), whilst the band gap due to the fractal perturbations occurs at the edge of the BZ. Therefore,
there must always be some allowed wave states (however narrow a band) between the two band gaps, and so no fractal perturbations of this type will be able to widen the original band gap by introducing an extra band gap that overlaps with the original one.

### 7.3.2 Fractal heterostructures

The above analysis suggests that fractal systems can be of no use in widening a single band gap. However, they have been shown to introduce extra band gaps at frequencies other than the original band gaps of the Bragg stack. It is possible to use these
extra band gaps to construct more efficient series heterostructures that achieve a wider band gap for a shorter structure length than could be achieved with equivalent periodic systems alone.

Consider again the original fractal system (Fig. (7.7)) where the fractal perturbations were centred in every layer and had a thickness of half the layer thickness. The transmission coefficient displayed two clear band gaps (centred at approximately 1 kHz and 2 kHz), separated by a region of high transmission. If this structure is placed in series with a periodic structure which has a band gap covering the central transmission region, a wide band gap should result. This is achieved using a Bragg stack structure with a central frequency of 1.5 kHz. The resulting transmission coefficient for the individual structures and the heterostructure is shown in Fig. (7.10).

Good overlap is seen between the stop bands of the two systems, although ideally a slightly greater degree of overlap would be desired. Correspondingly, the heterostructure exhibits a wide stop band, although with some very small peaks in transmission corresponding to the edges of the original band gaps. These small peaks could be removed by a slight increase in the ratio between the two areas of the main structure, \( \eta \), as this would serve to widen the original band gaps slightly. Note, the wide band gap in this case (approximately 1.4 kHz) has been achieved with only two 8 period structures.

![Figure 7.10](image)

**Figure 7.10:** (a) The intensity transmission coefficients for an 8 period Bragg stack with a centre frequency of \( f_c = 1 \) kHz modified by fractal perturbations at the centre of each layer (with \( \delta S = S_1/4 \) and \( \delta a = 2a/3 \)), and an 8 period Bragg stack with \( f_c = 1.5 \) kHz. All structures are constructed in the standard test arrangement. (b) The intensity transmission coefficient for a heterostructure formed from the combination of these systems in series, exhibiting a wide flat stop band, although with very small peaks at the overlap between the bands.
whereas to achieve this with only periodic structures over the same frequency range would require three 8 period structures. However, to achieve this reduction in required length, a more complex structure (in terms of variation of area) is required.

### 7.4 Genetic algorithms

The above sections have all dealt with the transmission properties of well defined structures based on periodic or quasi-periodic sequences. It is also possible to use a numerical optimisation to determine a structure layer sequence (thickness of perturbation and spacing layers of the structure) which has optimal (minimum) transmission over a given frequency range. In this way, the structure is not constrained to any particular pattern of lengths or sequences of perturbations, and using this method, it may be possible to achieve a wide band gap structure using a smaller thickness or number of layers than has been achieved above.

The numerical optimisation considered in this work is a genetic algorithm (GA), which is an algorithm that uses evolutionary techniques to optimise a particular problem. A detailed description of the GA is beyond the scope of this work, and only a brief introduction to the important properties and parameters of the algorithm is given here. The interested reader is referred to Ref. [52] and the references therein.

The GA process can be represented as follows:

1. **Initialisation** – Generate a random population of a given size.
2. **Fitness** – Calculate the fitness of each population member using some cost parameter. If all members of the population have equally good fit, the solution has found a minimum error.
3. **Evolution** – Select a number of members of the population to die and select parents to breed to create replacement members. Allow some mutation here.
4. **Loop** – Goto step 2 with new population.

These main sections are described briefly here.
Initialisation

Initially, the size of the population, number of individuals to die per cycle and the mutation rate are specified. The population of the specified size is then randomised. In this case, the starting population is chosen to have (random) values between ±1 for each coefficient. A nominal centre frequency, $f_c$, is chosen which defines a length $\ell$ equal to a quarter-wavelength at $f_c$. A positive population value, $p$, represents a $B$ section of structure (perturbation) with length $\ell = |p|$, whilst a negative population value, $n$, represents an $A$ section of structure (spacing) with length $\ell = |n|$. It is possible to include the best solution from a previous run as a starting member of the population.

Fitness

A cost parameter is required which quantifies the deviation of the transmission properties of the structure from the desired response. Assuming the goal is to achieve a wide stop band over the frequency range of the model, a simple example is the mean of the transmission coefficient in the range considered

$$\varepsilon = \frac{1}{M} \sum_{m=1}^{M} (\alpha_t)_m$$  \hspace{1cm} (7.1)

where $(\alpha_t)_m$ is the transmission coefficient of each frequency sample of a given member of the population in the genetic algorithm, and $M$ samples are considered in the frequency range investigated. The minimum, maximum and mean fitness values are stored. For this application, a tolerance of the maximum transmission value allowed in the range is introduced. If the transmission at any frequency rises above this value, the fitness value is increased, making the population much less fit, and therefore less likely to survive.

Evolution

The decisions for breeding and dying are made by forming a cumulative probability distribution, $c_n$. Population members with high fitness values (higher transmission in the frequency range) have large steps in the ‘dying’ $c_n$. The exact population member to die is chosen by generation of a random number, but large steps in $c_n$ lead to a higher probability of that individual dying. The individual with the best fitness is not able to die. Similarly, those with the best (lowest) fitness values are more likely to be chosen
to breed, with the least fit not able to breed. It is possible for the individual that has
died to die again, or a parent to breed with itself. These represent coding limits of the
model, which slightly reduce its efficiency, but are not large limitations.

Once the parents have been chosen, they must breed a new offspring to replace the
population member that has died. The genes of the new offspring are chosen to be
randomly distributed between the parent genes. For each gene, this is described by

\[ g_c(n) = g_{p1}(n) + r\Delta \] (7.2)

where \( g_c(n) \) is the child’s \( n^{th} \) gene, \( g_{p1}(n) \) is the first parent’s \( n^{th} \) gene, \( r \) is a random
number between zero and unity (generated from a rectangular probability distribution)
and \( \Delta \) is the difference between the parents’ gene values

\[ \Delta = g_{p2}(n) - g_{p1}(n). \] (7.3)

Some genes can also mutate. The number of genes that mutate is determined by
dividing the mutation rate by a random number, and rounding the resulting number to
the closest lower integer. Therefore, a high mutation rate will have a higher probability
of leading to a higher number of genes mutating. For each mutation, the gene that
mutates is again chosen randomly, although the same gene is not allowed to mutate
twice. Therefore, the maximum number of mutations is limited to the number of genes,
and if the number that mutate is equal to this number, all genes are mutated. The
mutated gene can be described by

\[ g_c'(n_m) = g_c(n_m) \pm (r_g + |\Delta|) \] (7.4)

where \( n_m \) is the number of the gene that mutates, \( \Delta \) is given by Eqn. (7.3) and \( r_g \) is a
random number generated from a normal distribution with mean 0 and a given variance
that can be changed in the model to affect the size of mutation. Thus, the mutation
has a magnitude that is normally distributed with a mean of \(|\Delta|\) and specified variance.
The mutation can either add to or subtract from the previous gene value, and this is
chosen randomly. The fitness is calculated for this new member of the population.
Looping

The dying and breeding of genes continues until either the solutions converge or the number of iterations exceeds a specified value. Once one of these criteria is fulfilled, the population member with the best fitness is obtained (all population members will have the same genes if the solution has truly converged). The variation of minimum, maximum and mean fitness values is plotted, and the transmission resulting from the best member of the population is calculated and plotted. Finally, this solution is saved. If the optimum solution has not been found, the algorithm can be re-run, introducing the best final solution as a member of the initial population.

Many of the parameters outlined above must be optimised in order to achieve effective operation of a genetic algorithm. Some of the most important are

- Mutation rate;
- Mutation size;
- Gene breeding;
- Selection of individuals to die and breed;
- Starting population (seeding of the population);
- Size of population.

The genetic algorithm is therefore rather complicated to initialise, and there may be more efficient algorithms for the optimisation processes. However, the exact algorithm and optimisation is not of interest in this work. Rather, the predicted structures to achieve broad stop bands will be discussed below.

7.4.1 GA results

As evident from the discussions in the previous chapters, obtaining a wide band gap with a small number of layers is difficult. Indeed, even with numerical optimisation, it proved impossible to achieve any advantage on the periodic systems with the same parameters as used above. However, increasing the ratio of the area in the perturbed sections with respect to the ‘normal’ sections of waveguide to four, $\eta = \frac{S_1}{S_2} = 4$, allowed numerical solutions to be found with stop bands that are significantly wider than can
Table 7.1: The structure sequence predicted by two runs of the genetic algorithm specified to minimise transmission over the frequency range \(600 \text{ Hz} \leq f \leq 2000 \text{ Hz}\).

<table>
<thead>
<tr>
<th>Layer number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structure type</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>Thickness (m)</td>
<td>0.0505</td>
<td>0.0386</td>
<td>0.0022</td>
<td>0.0582</td>
<td>0.0202</td>
<td>0.0162</td>
<td>0.0047</td>
<td>0.0260</td>
<td>0.0019</td>
<td>0.0213</td>
</tr>
</tbody>
</table>

Table 7.2: The reduced structure sequence (created by combining adjacent layers of the same type in Table (7.1)) predicted by two runs of the genetic algorithm specified to minimise transmission over the frequency range \(600 \text{ Hz} \leq f \leq 2000 \text{ Hz}\).

<table>
<thead>
<tr>
<th>Layer number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structure type</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>Thickness (m)</td>
<td>0.0305</td>
<td>0.0191</td>
<td>0.0295</td>
<td>0.0154</td>
<td>0.0537</td>
<td>0.0192</td>
<td>0.0302</td>
</tr>
</tbody>
</table>

be achieved using periodic, quasi-periodic or heterostructure arrangements with similar parameters. Note, the ability of the GA to improve on the periodic and quasi-periodic cases for larger values of \(\eta\) whilst not being able to find more optimal solutions for smaller values of \(\eta\) may be due to the complicated error surface. The GA is attempting to find the global minima of the error surface. However, for smaller \(\eta\), the minima will not be as deep, and so the algorithm may more easily become trapped in local minima that do not have optimal transmission properties. As \(\eta\) is increased, the minima in the error surface corresponding to optimal transmission would be expected to become deeper, improving the chances of optimal (or at least better) solutions being found.

The GA was defined to minimise the transmission over the frequency range \(600 \text{ Hz} \leq f \leq 2000 \text{ Hz}\). The algorithm was run twice, the second time with the initial population seeded with the best solution from the first run. A high mutation rate that reduced with increasing iterations through the algorithm was required. 20 layers of any thickness were allowed in the solution, and each layer could be defined as either an \(A\) (‘normal’) or \(B\) (perturbation) layer. The algorithm’s structure solution is shown in Table (7.1). The structure prediction contains several regions where the same layer types are adjacent. These can therefore be reduced into single layers, and the resulting structure definition is shown in Table (7.2). The total structure length is therefore only 0.53 m. The resulting transmission coefficient for the structure is shown in Fig (7.11). Although some small peaks are present (within the tolerance of \((\alpha_{t})_{\text{max}} = 0.1\) specified in the GA) a wide
stop band results over the approximate range $600 \, \text{Hz} \leq f \leq 2000 \, \text{Hz}$, which is much wider than could be achieved with periodic or quasi-periodic systems of an equivalent length. If the small peaks in transmission within the stop band can be tolerated for the application of the structure, this method of solution represents an efficient way to optimise a structure to have low transmission over a broad band for small lengths of structure.

However, of greater interest than this is whether any insight can be gained into the mechanisms for the band gap formation in non-periodic media. In order to help elucidate these mechanisms, the (non-integer) auto-correlation function of the structure sequence is shown in Fig. (7.12). The ACF shows reasonably strong periodicities between 1 and 2 layers and between 2 and 3 layers. Other examples of structures generated by the GA (they will not always be the same due to the presence of many local minima in the error surface, in which the algorithm can get trapped) generally have an ACF with a strong periodicity between 2 and 3 layer delays. This would therefore seem to be a potential feature responsible for the stop band formation.

Comparing these ACFs to those of, for example, the Fibonacci systems, shows that again, in these cases, strong periodicities are seen at 2 and 3 layer delays. The exact periodicity and layer construction will ultimately be responsible (due to the exact sequence of reflections and hence interference) for the transmission through the structure. These
results do, however, suggest that periodicities of the order 2 to 3 are required in order to achieve stop bands within the material. This includes the periodic systems (with periodicities of 2) which give rise to very well defined band gaps. This also includes the Fibonacci systems (with strong periodicities of 3) which give rise to pseudo-band gaps interrupted by significant defect peaks. The structures determined by the GA seem to fall somewhat between these two examples, with strong periodicities mid-way between two and three delays. This therefore results in wide stop bands that are not interrupted significantly by defect peaks. However, the bands are not as well formed as for the periodic or Fibonacci (i.e. deterministic sequence based) systems.

**Figure 7.12:** The (non-integer) auto-correlation function of the structure sequence described in Table (7.2).
Chapter 8

Discussions

The previous chapters have detailed the transmission properties of periodically and quasi-periodically modulated waveguides, and have shown that reject bands exist with varying degrees of width and depth. In all cases, these stop bands form as a result of scattering of incident acoustic waves and the related interference. The resulting transmission, however, is quite different in the different cases. This chapter aims to provide a brief review of the properties of the systems, comparing and contrasting them for use in potential applications.

Simple periodic systems result in very well defined stop bands. As the number of periods is increased, full band gaps form in the transmission, and these are very flat with steep sides. These are therefore suitable structures for use when single well defined band-stop filters with high roll-off rates are required. The centre frequency of the band gap can be tuned by altering the thickness of the layers and the width can be adjusted by varying both the central frequency and the variation of area between the layers (i.e. changing the impedance discontinuity between the layers). The previous analysis showed that closed-form analytical expressions can be obtained for band gap position and width, making this system very powerful when specific band gap positions are required. Furthermore, at higher frequencies, fairly wide gaps can be achieved. These systems therefore have potential use in, for example, forced air heating systems to prevent transmitted machine noise. Of course, in a system such as this with sharp discontinuities in area, care must be taken to prevent the generation of large amounts of turbulent noise. In addition to this, however, the properties of the system could easily be tuned to give a narrow bandwidth, and hence this system could be implemented as
a narrow-band band stop filter.

Alternatively, it is possible, by introducing a defect to the periodic structure, to create a narrow band transmission peak in the band gap with high Q-factor and extinction ratio. The position of this peak can be varied by varying the size of the defect. This allows such structures to form very narrow transmission filters, although, as demonstrated, extreme care must be taken with the experimental configuration to maintain the high Q-factors.

In contrast to the periodic systems, the transmission through the quasi-periodic systems was seen to be significantly more complicated. For the Fibonacci systems, although pseudo-band gaps were seen to form (which can be fairly well defined with steep edges and low transmission for high order structures) these were separated by a series of significant defect peaks spaced over a range of frequencies about the centre of the band gap for the equivalent periodic structure. The transmission between the gaps is also complicated, not displaying the simple oscillatory behaviour of the periodic systems. This is due to the more complicated sequence of reflections giving rise to the transmission properties of the structure. Alternative quasi-periodic sequences, with higher degrees of aperiodicity than the Fibonacci structures, showed more complex variations in transmission with less well defined pseudo-band gaps than the Fibonacci structures. In isolation, therefore, the quasi-periodic structures would seem to be of limited practical use.

Combining several different structures together to form heterostructures was seen to widen the band gaps observed in the transmission. In this case, it was possible to achieve good results with Fibonacci structures, if they were combined with periodic structures containing a band gap between the two pseudo-gaps of the Fibonacci structure. Further examples were shown which would be expected to give rise to broad omnidirectional bands. Alternatively, several periodic structures could be combined. Both structures are therefore suitable to achieve wide band gaps in this way. In principle, the gap can be made arbitrarily wide by including more and more structures into the heterostructure, although practical considerations would limit the number of structures that could be included. These systems could be used to provide effective filtering over a broad band of frequencies, dependent on the physical length of structure allowed.

A potentially more effective method of obtaining wider band gaps is to combine a periodic system with a fractal type system. Although the extra band gaps imposed by
fractal type perturbations cannot in themselves lead to wider band gaps from single structures, combining these fractal structures (with the extra, albeit separated in frequency, band gaps) with suitable periodic structures (where the periodic structure has a band gap covering the region of transmission between the fractal band gaps) wide band gap heterostructures can be obtained utilising smaller total thicknesses of structure than if just using periodic structures. This is, in general, desired for physical applications. The cost of this approach is a more complicated structure to construct (more variations of area due to the fractal nature). Alternatively, a numerical optimisation can be used to minimise transmission over a specified frequency range. Although this can give low transmission over a wide frequency range for short structures, the transmission was seen to still exhibit small peaks in the relevant range. This type of approach could therefore only be used in applications where these small peaks are insignificant.

Clearly, therefore, although all features seen in the transmission of acoustic waves through area modulated structures result from the same fundamental phenomenon of wave interference, the resulting transmission properties are highly dependent on the exact configuration of layers within the structure. Correspondingly, the structure type suitable for use in practical applications is based on a combination of the desired transmission properties of the structure, the degree of tolerance allowed in this transmission and the physical constraints (most importantly length and degree of area variation allowed) for the structure.
Chapter 9

Conclusions and further work

9.1 Conclusions

This work has considered the acoustic properties of periodically and quasi-periodically modulated waveguides. Both theoretical and experimental studies were made. The theoretical calculations utilised the transfer matrix method, and the details of this and a Bloch wave analysis were introduced. The experimental configuration and methods of calculation of the transmission coefficient and Bloch wavenumber from the experimental data were described.

Initially, locally periodic systems were investigated, and the transfer matrix approach was adapted to give closed-form theoretical expressions for the positions and size of band gaps within the structure for infinitely periodic systems. These band gaps were seen to be well approximated by locally periodic systems with only approximately 10 periods. For Bragg stacks (where the perturbation and spacing lengths are equal) the band gaps were seen to occur at the Brillouin zone (BZ) edge, but not at its centre. The centre frequency of the first band gap is equal to the frequency of a wave such that the period length of the structure is equal to half a wavelength of the wave. Increasing the period length was therefore seen to decrease the centre frequency of the band gaps, but also to decrease their width.

The width of the band gap was also seen to be controlled by the variation between the two areas of the structure due to the resulting change in impedance discontinuity. Having non-equal perturbation and spacing lengths (but still in a locally periodic system) was seen to split the band gaps and result in a more complicated variation of the transmission
with frequency.

Experimental results were considered for all the above examples, which showed approximate agreement in trend (but with small frequency shifts in the features) as well as an extra dip in observed transmission not predicted by the theoretical calculations. In the previous theory, the radiation impedance acting at a change in area had been neglected. Including this mass loading, and performing an empirical fit to the measured data to determine its magnitude, provided an excellent agreement between the theoretical and experimental data. Experimental band structure data was presented to indicate how the band gap opens up at the BZ edge as more periods are included in the structure.

A defect was introduced into the locally periodic Bragg stack, where the length of the centre section of the structure was altered. It was shown that this (theoretically) introduced a narrow-frequency perfect transmission peak within the band gap, with very high Q-factor and high extinction ratio. For defect lengths larger than the original section in the Bragg stack (so-called donor-type defects) the defect peak was seen to enter from the top of the band gap, moving to lower frequencies as the length was increased, occurring at the centre frequency of the gap when the defect was exactly twice the length of the original Bragg stack layer length. The reverse was seen for defect lengths shorter than the original section in the Bragg stack (so-called acceptor-type defects) where the defect peak was seen to enter from the bottom of the band gap, moving to higher frequencies as the length was decreased. The defect peak was seen to occur at the centre of the band gap when the defect length was reduced to zero. The Q-factor and extinction ratio of the defect peak were increased for structures with more periods and also with the proximity of the defect peak to the centre of the gap, due to the increased spatial confinement of the defect mode in these cases.

Experimental results showed good agreement with the theoretical predictions, especially in positioning of the defect peak. However, the magnitude of transmission (and hence Q-factor and extinction ratio) measured for the defect peak was lower than predicted by theory due to experimental errors in the construction providing a small variation in the coupling frequency to the defect mode, and hence a reduction in transmission at the defect peak frequency. This reduction was especially prevalent when the defect mode was closer to the centre of the gap and when higher numbers of periods
were used due to the higher spatial confinement of the defect mode in these cases.

Structures based on quasi-periodic systems were considered. In particular the Fibonacci, Thue-Morse and maximum length sequences were considered. The MLS signal shows the highest degree of aperiodicity and the Fibonacci the lowest. For the Fibonacci sequences, pseudo-band gaps were seen to develop spaced around the equivalent Bragg stack band gap centre frequency. These pseudo-gaps were separated by large multiple defect peaks, due to the large number of double spacing layers in the structure. Higher order sequences were seen to give rise to deeper, more sharply defined pseudo-band gaps. A self-similar structure was observed in the central defect peaks, which has been used previously as an indication of localization. The MLS structures showed no discernable band gaps, due to the lack of periodicity in the structure. The Thue-Morse system showed smaller gaps than the Fibonacci system with a larger number of peaks in the transmission, consistent with its degree of aperiodicity being between that of the other two structure types. For the Thue-Morse and Fibonacci systems, periodicities of 2 and 3 layers were judged to be important in the generation of pseudo-band gaps. In all cases, the experimental results showed good general agreement with the theoretical predictions when mass loading was included.

Finally, heterostructures, where one or more structure type is combined into a composite structure, were investigated with a view to widening the band gap. Combining periodic structures was seen to be suitable for increasing the band gap width, and by choosing suitable periodic structures the band gap could be made arbitrarily wide, bounded only by the physical constraints on the structure. Fibonacci structures were also shown to be suitable when a periodic structure was used to compensate for the defect peaks in the centre of the pseudo-gaps.

Fractal type systems were also considered where extra area perturbations were introduced to the main structure. Where an extra perturbation was included in the same relative position in each layer, an extra band gap was seen to occur at double the original band gap centre frequency due to the additional periodicity of the fractal structure. However, this was seen to suppress the original band gap width somewhat. Introducing fractal type perturbations with a periodicity incommensurate with that of the main structure allowed this band gap to move towards the original band gap. However, it was shown that the original band gap in this modified system opens up at the centre of
the BZ whereas this fractal band gap opens up at the edge of the BZ and so a transmission band (no matter how narrow) must always exist between the two band gaps, meaning this cannot be used to extend the initial band gap width. However, due to the extra band gaps introduced by the original fractal structures, these can be effectively combined with periodic structures to form a heterostructure with a wide band gap for relatively short structure lengths.

Finally, a genetic algorithm was employed to minimise the transmission over a given range by determining the optimum sequence and thickness of layers. Using this approach, it was possible to obtain a wide band gap using only 7 layers, but several small transmission peaks existed within this band gap that may be problematic in practical applications. Of greater interest was that this system showed periodicity components between two and three layer delays. This, in conjunction with the periodic and quasi-periodic results, suggests that large band gaps result when the structure is based on sequences with significant periodicity components between 2 and 3 layer separations.

### 9.2 Suggestions for further work

There are several possible extensions to this work. Firstly, the heterostructure systems were considered only theoretically due to limitations on the experimental configuration available. It would be of interest to confirm these theoretical results with experimental data. It would be of particular interest to experimentally measure the system predicted by the genetic algorithm in order to investigate the implications of experimental tolerances and variations on the measured response. It would also be of interest to consider further the quasi-periodic systems to gain further insight into the exact causes of the pseudo-band gaps with relation to the periodicities of the structures.

A host of other interesting phenomena are also expected for one-dimensional phononic crystals, such as tunnelling of forbidden pulses, change in group velocity of the propagating pulse and negative refraction of incident waves. These could be investigated using the model system introduced in this work, in order to gain understanding of the phenomena in a one-dimensional case, before higher dimensional structures are considered.
Appendix A

Transfer matrix relating pressure and volume velocity

In this appendix, a general relation shall be derived governing the relationship between the acoustic pressure and volume velocity at either end of a waveguide of constant cross-section, $S$, but with undefined terminating conditions.

Consider an air-filled waveguide of length $L$ and cross-section $S$ represented schematically in Fig. (A.1). The transfer matrix relating the acoustic pressure and volume velocity at $x = 0$ ($\Psi(0), U(0)$) to those at $x = L$ ($\Psi(L), U(L)$) can be represented

$$
\begin{pmatrix}
\Psi(0) \\
U(0)
\end{pmatrix} =
\begin{pmatrix}
A(\omega) & B(\omega) \\
C(\omega) & D(\omega)
\end{pmatrix}
\begin{pmatrix}
\Psi(L) \\
U(L)
\end{pmatrix}
$$

(A.1)

**Figure A.1:** A schematic representation of a waveguide of length $L$ used in the transfer matrix determination.
so that

\[ \Psi(0) = A(\omega) \Psi(L) + B(\omega) U(L) \]  
\[ U(0) = C(\omega) \Psi(L) + D(\omega) U(L). \]  

The ABCD coefficients may be calculated using the relations

\[ A(\omega) = \left. \frac{\Psi(0)}{\Psi(L)} \right|_{U(L)=0} \]
\[ B(\omega) = \left. \frac{\Psi(0)}{U(L)} \right|_{\Psi(L)=0} \]
\[ C(\omega) = \left. \frac{U(0)}{\Psi(L)} \right|_{U(L)=0} \]
\[ D(\omega) = \left. \frac{U(0)}{U(L)} \right|_{\Psi(L)=0} \]

which is equivalent to solving for the continuity of pressure and volume velocity at the boundaries of the waveguide. It is assumed that there exists a right and left travelling plane wave in the waveguide and so the total pressure in the waveguide is given by

\[ \Psi(x, t) = p_+ e^{ij(\omega t - kx)} + p_- e^{ij(\omega t + kx)} \]  

and application of Euler’s equation

\[ u = -\frac{1}{\rho_0} \int \frac{\partial \Psi}{\partial x} \, dt \]

gives the particle velocity

\[ u(x, t) = \frac{1}{\rho_0 c} \left( p_+ e^{j(\omega t - kx)} - p_- e^{j(\omega t + kx)} \right) \]  


giving a volume velocity

\[ U(x, t) = \frac{S}{\rho_0 c} \left( p_+ e^{j(\omega t - kx)} - p_- e^{j(\omega t + kx)} \right) \]  

For \( U(L) = 0 \),

\[ \frac{S}{\rho_0 c} \left( p_+ e^{j(\omega t - kL)} - p_- e^{j(\omega t + kL)} \right) = 0 \]

\[ \Rightarrow p_+ e^{-jkL} = p_- e^{jkL} \]

\[ \Rightarrow \frac{p_+}{p_-} = e^{2jkL}. \]  

Similarly, for $\Psi(L) = 0$,

$$p_+ e^{j(\omega t - kL)} + p_- e^{j(\omega t + kL)} = 0$$

$$\Rightarrow \frac{p_+}{p_-} = -e^{2jkL}. \quad (A.8)$$

The above relations can then be used to calculate the ABCD coefficients, as shown here.

$$A(\omega) = \left. \frac{\Psi(0)}{\Psi(L)} \right|_{U(L)=0}$$

$$= \left. \frac{(p_+ + p_-)e^{j\omega t}}{(p_+ e^{-jkL} + p_- e^{jkL})e^{j\omega t}} \right|_{U(L)=0}$$

$$= \left. \frac{p_+}{p_-} + 1 \right|_{U(L)=0}$$

and substitution of Eqn. (A.7) gives

$$A(\omega) = \frac{e^{2jkL} + 1}{e^{jkL} + e^{jkL}}$$

$$= \frac{1}{2} \left( e^{jkL} + e^{-jkL} \right)$$

$$= \cos(kL). \quad (A.10)$$

Similarly,

$$B(\omega) = \left. \frac{\Psi(0)}{U(L)} \right|_{\Psi(L)=0}$$

$$= \left. \frac{S}{\rho_0 c} \frac{(p_+ + p_-)e^{j\omega t}}{(p_+ e^{-jkL} - p_- e^{jkL})e^{j\omega t}} \right|_{\Psi(L)=0}$$

$$= \left. \frac{\rho_0 c}{S} \frac{p_+}{p_-} + 1 \right|_{\Psi(L)=0}$$

and substitution of Eqn. (A.8) gives

$$B(\omega) = \frac{\rho_0 c (-e^{2jkL} + 1)}{S}$$

$$= \frac{\rho_0 c}{2S} \left( e^{jkL} - e^{-jkL} \right)$$

$$= j \frac{\rho_0 c}{S} \sin(kL). \quad (A.12)$$
Similarly,

\[
C(\omega) = \left. \frac{U(0)}{\Psi(L)} \right|_{U(L)=0} = \frac{S \rho_0 c (p_+ - p_-) e^{j\omega t} \left|_{U(L)=0} \right.}{(p_+ e^{-jkL} + p_- e^{jkL}) e^{j\omega t}} = \frac{S}{\rho_0 c} \frac{p_+ - 1}{p_+ e^{-jkL} + e^{jkL}} = \frac{S}{2\rho_0 c} (e^{jkL} - e^{-jkL}) = j \frac{S}{\rho_0 c} \sin(kL) \tag{A.13}
\]

and,

\[
D(\omega) = \left. \frac{U(0)}{U(L)} \right|_{\Psi(L)=0} = \frac{S \rho_0 c (p_+ - p_-) e^{j\omega t} \left|_{\Psi(L)=0} \right.}{(p_+ e^{-jkL} - p_- e^{jkL}) e^{j\omega t}} = \frac{S}{\rho_0 c} \frac{p_+ - 1}{p_+ e^{-jkL} - e^{jkL}} = \frac{1}{2} (e^{jkL} + e^{-jkL}) = \cos(kL). \tag{A.14}
\]

Substitution of these relations into Eqn. (A.1) gives the general transfer matrix expression

\[
\begin{pmatrix}
\Psi(0) \\
U(0)
\end{pmatrix} = \begin{pmatrix}
\cos(kL) & j \frac{\rho_0 c}{S} \sin(kL) \\
j \frac{S}{\rho_0 c} \sin(kL) & \cos(kL)
\end{pmatrix} \begin{pmatrix}
\Psi(L) \\
U(L)
\end{pmatrix} \tag{A.15}
\]

where \( \frac{\rho_0 c}{S} \) is the acoustic impedance of the waveguide of length \( L \) for which the analysis applies.
Appendix B

Power of a unimodular matrix

The appendix demonstrates a closed-form solution to the arbitrary power of a unimodular matrix, following the approach of Griffiths and Steinke [23]. Consider a $2 \times 2$ unimodular matrix

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \quad (B.1)$$

whose characteristic equation is given by

$$\det(P - \lambda I) = 0 \quad (B.2)$$

where $I$ is the identity matrix and $\lambda$ is an eigenvalue of $P$. Eqn. (B.2) can be expanded to give

$$\lambda^2 - \lambda(P_{11} + P_{22}) + (P_{11}P_{22} - P_{21}P_{12}) = 0 \quad (B.3)$$

or

$$\lambda^2 - \lambda \text{Tr}(P) + \det(P) = 0. \quad (B.4)$$

Letting $\xi = \frac{1}{2} \text{Tr}(P)$ and recalling that $\det(P) = 1$ as the matrix is unimodular, Eqn. (B.4) becomes

$$\lambda^2 - 2\lambda\xi + 1 = 0. \quad (B.5)$$

The Cayley-Hamilton theorem states [53] that any matrix satisfies its own characteristic equation, and so

$$P^2 - 2P\xi + I = 0. \quad (B.6)$$
Thus,

\[ P^2 = 2P\xi - I \]  

and so any higher combination of \( P \) can be (iteratively) reduced to a linear combination of \( P \) and \( I \)

\[ P^N = PU_{N-1}(\xi) - IU_{N-2}(\xi) \]  

where \( U_N(\xi) \) is some polynomial of order \( N \) in \( \xi \). Treating Eqn. (B.8) as a recursion relation, and letting \( N \to (N + 1) \) gives

\[ P^{N+1} = PU_N(\xi) - IU_N - 1(\xi) \]  

whilst multiplying Eqn. (B.8) by \( P \) and substituting for \( P^2 \) from Eqn. (B.7) gives

\[ P^{N+1} = (2P\xi - I)U_{N-1}(\xi) - PU_{N-2}(\xi). \]  

These two expressions must be equivalent giving

\[ U_N(\xi) - 2\xi U_{N-1}(\xi) + U_{N-2}(\xi) = 0 \]  

which satisfies the recursion relation for Chebyshev polynomials of the second kind [54, ch. 22] and can be represented in trigonometric form as [55]

\[ U_N(\xi) = \frac{\sin[(N + 1)\gamma]}{\sin \gamma} \]  

where

\[ \gamma = \cos^{-1}(\xi). \]  

Therefore, the \( N \)th power of the matrix \( P \) can be calculated from Eqn. (B.8) with the polynomial functions defined by Eqns. (B.12) and (B.13).
Appendix C

Radiation load of a baffled piston

Consider a baffled plane circular piston of radius $a$ and surface velocity $u = \hat{u}e^{j\omega t}$. Following the derivation of Kinsler et al. [4], the pressure $dp$ produced at an elemental area $dS'$ due to motion of an elemental area $dS$ (which acts as a baffled monopole source)

$$dp = j\rho_0 c \frac{k}{2\pi r} e^{j(\omega t - kr)} = j\rho_0 c \frac{k}{2\pi r} \hat{u} e^{j(\omega t - kr)} dS$$  \hspace{1cm} (C.1)

can be integrated over the surface of the piston to give the total pressure at $dS'$

$$p = j\rho_0 c \hat{u} \frac{k}{2\pi} \int \frac{1}{r} e^{j(\omega t - kr)} dS.$$  \hspace{1cm} (C.2)

The radiation impedance is given by the ratio of the total force on the piston due to the pressure, $f$, to the surface velocity

$$Z_{rad} = \frac{f}{u} = \frac{\int p dS'}{u} = j\rho_0 c \frac{k}{2\pi} \int \int \frac{1}{r} e^{j(\omega t - kr)} dS dS'.$$  \hspace{1cm} (C.3)

The calculation of this integral is beyond the scope of this work, but the result is given [4] by

$$Z_{rad} = \rho_0 c S [R_1(2ka) + jX_1(2ka)]$$  \hspace{1cm} (C.4)

where $S$ is the surface area of the piston of radius $a$, the piston resistance function

$$R_1(x) = 1 - \frac{2J_1(x)}{x} = \frac{x^2}{2.4} - \frac{x^4}{2.4^2.6} + \frac{x^6}{2.4^2.6^2.8} - \cdots$$  \hspace{1cm} (C.5)
where \( J_1(x) \) is the Bessel function of the first kind, and the piston reactance function

\[
X_1(x) = \frac{2H_1(x)}{x} = \frac{4}{\pi} \left[ \frac{x}{3} - \frac{x^3}{3^2 \cdot 5} + \frac{x^5}{3^2 \cdot 5^2 \cdot 7} - \ldots \right]
\]  

(C.6)

where \( H_1(x) \) is the first order Struve function. In the low-frequency limit \( (ka \ll 1) \), Eqns. (C.5) and (C.6) can be approximated

\[
R_1(x) \approx \frac{x^2}{8}, \quad X_1(x) \approx \frac{4x}{3\pi}
\]

giving the approximate radiation load impedance

\[
Z_{rad} \approx \rho_0 c S \left( \frac{(ka)^2}{2} + j \frac{8}{3\pi} (ka) \right).
\]  

(C.7)

Eqn. (C.7) is in the form of a radiation (mechanical) impedance and so can be converted to an acoustic impedance by dividing by \( S^2 \) giving the radiation impedance (at low frequencies) in acoustic units

\[
Z_{A_rad} \approx \frac{\rho_0 c}{S^2} \left( \frac{(ka)^2}{2} + j \frac{8}{3\pi} (ka) \right).
\]  

(C.8)
Appendix D

MATLAB models

This Appendix contains the code for the MATLAB models used in this work, and the related functions.

D.1 Main modulated waveguide model

%-------------------------------------------------------------------------
%modulated_waveguide.m models the acoustic properties of a waveguide with
%periodic and quasi-periodic changes in area
%-------------------------------------------------------------------------
%Philip D. King
%Research project, MSc Audio Acoustics
%University of Salford, 2006.

clear all
close all
clc

%Initial specifications%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Physical constants of medium (assuming T=20degC, p=latm)
c = 343; %Speed of sound in air (ms^-1)
rho0 = 1.21; %Density of air (Kgm^-3)

%Frequency range of model
f = linspace(0,3000,3001);
omega = 2*pi*f; %Angular frequency range
k = (2*pi*f)/c; %Wavenumber in air

%Choice of model options
%Specify choice as zero (allows checks that correct selections made)
choice_p = 0;
choice_h = 0;

%Display periodicity options to screen
disp('Waveguide periodicity:');
disp('------------------------------------------');
disp('(1) Total periodicity');
disp('(2) Total periodicity with a single defect');
disp('(3) Quasi-periodicity (Fibonacci)');
disp('(4) Quasi-periodicity (MLS sequence)');
disp('(5) Quasi-periodicity (Thue-Morse sequence)');
disp('(6) Symmetric Fibonacci system');
disp('(7) Fractal system');
disp('(8) Fractal type Fibonacci system');
disp('(9) Heterostructure system');
while choice_p < 1 | choice_p > 10 %Ensure that choice is made from list of options
    choice_p = input('Choose periodicity constraints: '); %Select structure periodicity
    if choice_p < 1 | choice_p > 10 %Warn choice is not valid
        disp('Choice not valid - Please choose from listed options')
    end
end

%Default no heterostructure options
het_no = 1; %Define number of structure types in system
het = 0; %Set heterostructure flag to off

switch choice_p
    case 9
        %Define the properties of the heterostructure
        het_no = input('Enter number of structure types in heterostructure: '); %Set a flag to 1 to indicate a heterostructure system
        het = 1; %Set choice to zero
        clc %Clear the screen
        %Display periodicity options for each section to screen
        disp('Waveguide periodicity:');
        disp('-----------------------------------------------------------');
        disp('(1) Total periodicity');
        disp('(2) Total periodicity with a single defect');
        disp('(3) Quasi-periodicity (Fibonacci)');
        disp('(4) Quasi-periodicity (MLS sequence)');
        disp('(5) Quasi-periodicity (Thue-Morse sequence)');
        disp('(6) Symmetric Fibonacci system');
        disp('(7) Fractal system');
        disp('(8) Fractal type Fibonacci system');
        disp('-----------------------------------------------------------');
        for c1 = 1:het_no
            while choice_h(c1) < 1 | choice_h(c1) > 8 %Ensure that choice is made from list of options
                %Select periodicity for each structure
                choice_h(c1) = input(['Choose periodicity constraints for structure ',num2str(c1), ': ']);
                if choice_h(c1) < 1 | choice_h(c1) > 8 %Warn choice is not valid
                    disp('Choice not valid - Please choose from listed options')
                end
            end
        end
end

%Choose whether to include mass loading effects
%Specify choice as zero (allows checks that correct selections made)
choice_ml = 0;
if choice_p ~= 7 & choice_p ~= 8 %Don't give choice if fractal systems considered
    %Display options to screen
    disp('Include mass loading effects?');
    disp('-----------------------------------------------------------');
    disp('(1) Yes');
    disp('(2) No');
    disp('-----------------------------------------------------------');
    while choice_ml < 1 | choice_ml > 2 %Ensure that choice is made from list of options
        choice_ml = input('Choose whether to include mass loading effects: '); %Select mass loading option
        if choice_ml < 1 | choice_ml > 2 %Warn choice is not valid
            disp('Choice not valid - Please choose from listed options')
        end
    end
end

%Run the calculations for the total number of structures (relevant if a
%heterostructure is being modelled
for str_no = 1:het_no
    %If a heterostructure is being modelled, define the section structure type
    if het == 1
        choice_p = choice_h(str_no);
    end
% Define all other choice values to zero to allow checks on choices to be made
choice_f = 0;
choice_m = 0;
choice_tm = 0;
choice_a = 0;

% Display current structure type to screen (for identification when
% constructing a heterostructure)
clc
disp('-----------------------------------------------')
switch choice_p
  case 1
disp('Totally periodic system');
  case 2
disp('Totally periodic system except for a single defect');
  case 3
disp('Quasi-periodic system (Fibonacci sequence)');
  case 4
disp('Quasi-periodic system (MLS sequence)');
  case 5
disp('Quasi-periodic system (Thue-Morse sequence)');
  case 6
disp('Symmetric Fibonacci system');
  case 7
disp('Fractal system');
  case 8
disp('Fractal type Fibonacci system');
end
disp('-----------------------------------------------')

% Display section length options to screen
disp('Section lengths:');
disp('-----------------------------------------------');
disp('(1) A and B sections have same length (equal to half the period)');
disp('(2) A and B sections have different lengths');
disp('-----------------------------------------------');
while choice_a < 1 | choice_a > 2 % Ensure that choice is made from list of options
  choice_a = input('Choose section length constraints: '); % Select input signal
  if choice_a < 1 | choice_a > 2 % Warn choice is not valid
    disp('Choice not valid - Please choose from listed options')
  end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Design parameters of the stack
% Request user to select design parameters

%Number of periods in the stack

n = input(['Enter number of periods for structure: ']);

% Design frequency (desired frequency of first stop band)
f_c = input(['Enter design frequency for structure: ']);

% Calculate remaining parameters

lambda_c = c/f_c; % Corresponding design wavelength
s = lambda_c/2; % Corresponding period length

switch choice_a
  case 1 % If half period length chosen
    a = s/4; % Half the length of the perturbation
  case 2 % If other length chosen
    a = s/8; % Half the length of the perturbation
end
l = s-(2*a); % Length of waveguide between perturbations

% Areas of each section
S1 = (54e-3)^2;
S2 = (38e-3)^2;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Transfer matrix specification

switch choice_p % Perform calculations based on waveguide being modelled
case 1 %If total periodicity modelled
%Generate transfer function matrix elements, Bloch phase and area ratio from function periodic_layers_tf.m
[P_N11 P_N12 P_N21 P_N22 xi eta] = periodic_layers_tf(n,k,a,s,S1,S2,rho0,c,omega,choice_ml);

case 2 %If a defect is included
%Define defect parameters
nd = floor((n+1)/2); %Number of the period containing the defect
if n/2 == floor(n/2) %If n is even
ad = a; %Half length of expansion in defect period
ld = 0.5*l; %Length of constriction in defect period
else %If n is odd
ad = a*0.5; %Half length of expansion in defect period
ld = l; %Length of constriction in defect period
end
sd = ld+(2*ad); %Defect period length
S2d = S2; %Allows different width for defect section
%Define three transfer matrices describing before, after and the actual defect propagation from function periodic_layers_tf.m
%Total transfer function matrix elements before defect
[P_A11 P_A12 P_A21 P_A22 xi eta] = periodic_layers_tf(nd-1,k,a,s,S1,S2,rho0,c,omega,choice_ml);
%Single period transfer matrix elements for defect layer
[P_D11 P_D12 P_D21 P_D22] = periodic_layers_tf(1,k,ad,sd,S1,S2d,rho0,c,omega,choice_ml);
%Total transfer function matrix elements after defect
[P_B11 P_B12 P_B21 P_B22] = periodic_layers_tf(n-nd,k,a,s,S1,S2,rho0,c,omega,choice_ml);
%Total transfer function matrix elements (concatenate three previous transfer matrices together)
%Concatenate the first two sections
[M11 M12 M21 M22] = concat(P_A11,P_A12,P_A21,P_A22,P_D11,P_D12,P_D21,P_D22);
%Concatenate the final section
[P_N11 P_N12 P_N21 P_N22] = concat(M11,M12,M21,M22,P_B11,P_B12,P_B21,P_B22);

case {3,4,5,6} %If quasi-periodic system is being modelled
%Choose quasi-periodic sequence order and define sequence
switch choice_p
  case {3,6} %If Fibonacci system being modelled
    %Display Fibonacci options to screen
    while choice_f < 2 | choice_f > 9 %Ensure that choice is made from list of options
      choice_f = input('Choose order of Fibonacci section (2 to 9): '); %Choose order of Fibonacci section (2 to 9)
      if choice_f < 2 | choice_f > 9 %Warn choice is not valid
        disp('Choice not valid - Please choose from given range')
      end
    end
    [FS S] = Fib(choice_f);
    %Make Fibonacci structure symmetric if required
    if choice_p == 6
      S = [S fliplr(S)]; %Makes this sequence symmetrical
    end
  case 4 %If MLS system being modelled
    %Display MLS options to screen
    while choice_m < 2 | choice_m > 20 %Ensure that choice is made from list of options
      choice_m = input('Choose order of MLS sequence (2 to 20): '); %Choose order of MLS sequence (2 to 20)
      if choice_m < 2 | choice_m > 20 %Warn choice is not valid
        disp('Choice not valid - Please choose from given range')
      end
    end
    [mls S] = MLS_layer(choice_m);
  case 5 %If Thue-Morse system being modelled
    %Display Thue-Morse options to screen
    while choice_tm < 1 | choice_tm > 10 %Ensure that choice is made from list of options
      choice_tm = input('Choose order of Thue-Morse sequence (1 to 10): '); %Choose order of Thue-Morse sequence (1 to 10)
      if choice_tm < 1 | choice_tm > 10 %Warn choice is not valid
        disp('Choice not valid - Please choose from given range')
      end
    end
    [Thue_Morse S] = Thue_Morse_layer(choice_tm);
disp('Choice not valid - Please choose from given range')

%Generate Thue-Morse layer sequence
[TM S] = Thue_Morse(choice_tm);

%Generate transfer function matrix elements, Bloch phase of simple periodic case and area ratio from function quasiperiodic_layers_tf.m
[P_N11 P_N12 P_N21 P_N22 xi eta] = var_thick_layers_tf(n,S,k,a,s,S1,S2,rho0,c,omega,choice_ml);

case 7
% If a fractal case is being considered
% Define fractal properties
fr = 2; % Order of fractal
delta_S = S1/4; % Variation in cross-section
% Generate transfer function matrix elements for n periods from function fractal.m
[P_N11 P_N12 P_N21 P_N22 xi eta] = fractal(n,k,a,s,S1,S2,delta_S,fr,rho0,c);

case 8
% If a fractal type FS case is being considered
% Display Fibonacci options to screen
while choice_f < 2 | choice_f > 9 % Ensure that choice is made from list of options
    disp('-------------------------------------------------');
    choice_f = input('Choose order of Fibonacci section (2 to 9): ');
    if choice_f < 2 | choice_f > 9 % Warn choice is not valid
        disp('Choice not valid - Please choose from given range')
    end
end

% Generate Fibonacci layer sequence
[FS S] = Fib(choice_f);
% Define fractal properties
delta_S = S1/4; % Variation in cross-section
fr = 2; % Order of fractal system
% Generate transfer function matrix elements for n periods from function fractal.m
[P_N11 P_N12 P_N21 P_N22 xi eta] = fractal2(n,k,a,s,S1,S2,delta_S,FS,rho0,c);

case 10
% Load in pre-defined structure thickness array where
% +ve value = length of B section
% -ve value = length of A section
load 'strt.mat'
S = strt; % Specify structure array

% Generate transfer function matrix elements, Bloch phase of simple periodic case and area ratio from function quasiperiodic_layers_tf.m
[P_N11 P_N12 P_N21 P_N22 xi eta] = var_thick_layers_tf(1,S,k,a,s,S1,S2,rho0,c,omega,choice_ml);

%------------------------------------------------------------------------
% If a heterostructure is being modelled, concatenate heterostructure matrices together to give total response of heterostructure
if het == 1
    if str_no == 1 % For first time through
        % Transfer matrix elements for first structure
        P_H11 = P_N11;
        P_H12 = P_N12;
        P_H21 = P_N21;
        P_H22 = P_N22;
    else % For subsequent runs
        % Concatenate into previous elements
        [P_H11 P_H12 P_H21 P_H22] = concat(P_H11,P_H12,P_H21,P_H22,P_N11,P_N12,P_W11,P_W22);
    end
end

%------------------------------------------------------------------------
% End the heterostructure loop
end

%------------------------------------------------------------------------
if het == 1
% Define total transfer matrix elements
P_N11 = P_H11;
P_N12 = P_H12;
P_N21 = P_H21;
P_N22 = P_H22;

%Set the structure choice parameter back to a heterostructure if required
choice_p = 9;
end

%--------------------------------------------------------------------------
%Calculate transmission and reflection (intensity) coefficients
%Transmission
alpha_t = abs(1./(0.5*(P_N11+(P_N12/(rho0*c/S1))+((rho0*c/S1)*P_N21)+P_N22))).^2;
%Reflection
alpha_r = abs((P_N11+(P_N12/(rho0*c/S1))-((rho0*c/S1)*P_N21)-P_N22)/(P_N11+(P_N12/(rho0*c/S1))+(rho0*c/S1)*P_N21)+P_N22)).^2;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Plot graphs of coefficients
%Define titles for reflection and transmission coefficient graphs
switch choice_p
  case 3
    title_refl = ('Intensity reflection coefficient, Fibonacci order ',num2str(choice_f),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
    title_trans = ('Intensity transmission coefficient, Fibonacci order ',num2str(choice_f),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
  case 4
    title_refl = ('Intensity reflection coefficient, MLS order ',num2str(choice_m),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
    title_trans = ('Intensity transmission coefficient, MLS order ',num2str(choice_m),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
  case 5
    title_refl = ('Intensity reflection coefficient, Thue-Morse order ',num2str(choice_tm),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
    title_trans = ('Intensity transmission coefficient, Thue-Morse order ',num2str(choice_tm),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
  case 6
    title_refl = ('Intensity reflection coefficient, Symmetric Fibonacci order ',num2str(choice_f),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
    title_trans = ('Intensity transmission coefficient, Symmetric Fibonacci order ',num2str(choice_f),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
  case 7
    title_refl = ('Intensity reflection coefficient, Fractal system (order ',num2str(fr),') ',num2str(n),', periods, f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
    title_trans = ('Intensity transmission coefficient, Fractal system (order ',num2str(fr),') ',num2str(n),', periods, f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
  case 8
    title_refl = ('Intensity reflection coefficient, Fractal system (order ',num2str(fr),') ',Fibonacci order, 'num2str(choice_f),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
    title_trans = ('Intensity transmission coefficient, Fractal system (order ',num2str(fr),') ',Fibonacci order, 'num2str(choice_f),', f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
  case 9
    title_refl = ('Intensity reflection coefficient, ',num2str(het_no),' element heterostructure');
    title_trans = ('Intensity transmission coefficient, ',num2str(het_no),' element heterostructure');
  otherwise
    title_refl = ('Intensity reflection coefficient, ',num2str(n),', periods, f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
    title_trans = ('Intensity transmission coefficient, ',num2str(n),', periods, f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta));
end

%Plot coefficients
figure
plot(f,alpha_r) %Plot reflection coefficient
xlabel('Frequency (Hz)') %x-axis label
ylabel('Reflection coefficient, \alpha_r') %y-axis label
axis([min(f),max(f),0,1.05]) %axes limits
grid on %Turn gridlines on
title(title_refl) %Add a title to the graph

figure
plot(f,alpha_t) %Plot transmission coefficient
xlabel('Frequency (Hz)') %x-axis label
ylabel('Transmission coefficient, $\alpha_t$')
axis([min(f), max(f), -0.05, 1])
grid on

title('Transmission vs frequency')

%Plot the cosine of the Bloch phase

figure
plot(xi, f)
xlabel('xi = \cos\gamma')
ylabel('Frequency (Hz)')
grid on

%Also plot limits for propagating waves

hold on
axis([min(xi) - 0.1, max(xi) + 0.1, min(f), max(f)])

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Show predicted total reflection/zero transmission bands for infinite
%perfectly periodic system on reflection/transmission plots

pred = zeros(size(f));

for each frequency where $|xi|>1$ (i.e. where there should be total
%reflection) set array to unity. This represents the reflection due to the
%stop bands

pred(find(abs(xi)>1)) = 1;

figure
plot(f, alpha_r)

xlabel('Frequency (Hz)')
ylabel('Reflection coefficient, $\alpha_r$')
axis([min(f), max(f), 0, 1.05])
grid on

%Do the same for transmission

figure
plot(f, 1-pred)

xlabel('Frequency (Hz)')
ylabel('Transmission coefficient, $\alpha_t$')
axis([min(f), max(f), -0.05, 1])
grid on

case 1
    %Calculate band structure for infinite perfectly periodic system
    K = (1/s)*acos(xi);
    %Select only the real values to plot
    Kbs = K(find(imag(K)==0))/(pi/s); %Band structure K values normalised to pi/period
    fbs = f(find(imag(K)==0)); %Corresponding frequency values

    xlim([-1.1, 1.1])
    xlabel('Bloch wavevector, K (normalised to $\pi/\Lambda$)')
    ylabel('Frequency, (Hz)')
grid on

%Further plots and calculations

switch choice_p
    %Consider when perfect periodicity is modelled
    case 1
        %Plot the wave vector vs omega for both positive and negative K
        plot(Kbs, fbs, 'bo', 'MarkerSize', 2, 'MarkerFaceColor', 'b');
        hold on
        plot(-Kbs, fbs, 'bo', 'MarkerSize', 2, 'MarkerFaceColor', 'b');
        hold off
        xlim([-1.1, 1.1])
        xlabel('Bloch wavevector, K (normalised to $\pi/\Lambda$)')
        ylabel('Frequency, (Hz)')
grid on

D MATLAB MODELS
MATLAB models

```matlab
% Plot real and imaginary components of Bloch wave-vector
figure
plot(real(K)/(pi/s),f,imag(K)/(pi/s),f)
xlabel('Bloch wavevector, K (normalised to \pi/\Lambda)') %X-axis label
ylabel('Frequency, (Hz)') %Y-axis label
grid on %Turn gridlines on
legend('Real','Imaginary') %Apply a legend

% Add a title to the graph
title(['Band structure, f_c = ',num2str(f_c),' Hz, \eta = ',num2str(eta)])

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Band gap width (and variation with \eta)
Delta_f = (c/(4*pi*a))*(pi-(2*asin(2*sqrt(eta)/(eta+1))));
% Display band gap to screen
disp(['Expected Band gap = ',num2str(Delta_f),' Hz'])

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Consider with varying \eta
eta_prime = linspace(1,200,199);
Delta_fp = (c/(4*pi*a))*(pi-(2*asin(2*sqrt(eta_prime)./(eta_prime+1))));
% Plot band gap variation
figure
plot(eta_prime,Delta_fp,eta,Delta_f,'r+','MarkerSize',10)
xlabel('\eta') %X-axis label
ylabel('Band gap, \Delta f (Hz)') %Y-axis label
grid on %Turn gridlines on

% Apply a title
title(['Variation of band gap with \eta (s = ',num2str(s),' m)'])

% Label actual band gap point
text(eta+0.01,Delta_f+100,
['\eta = ',num2str(eta)])

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Calculate the Q-factor of the defect peak if applicable
if choice_p == 5
f_d = f_c;
end

% Approximate FWHM of the peak (calculated using function fwhm.m)
[delta_f,OK] = fwhm(alpha_t,f,f_d);
```
%Calculate Q-factor
Q = f_d/delta_f;

%Display Q-factor to screen
disp('')
disp('------------------------------------------------------')
disp([\'Q-factor of defect transmission peak: ',num2str(Q)])

%If resolution not sufficient, warn of potential error
if OK == 0
disp('Note, result may be in error due to insufficient resolution')
end

disp('------------------------------------------------------')

%Calculate the autocorrelation of the quasi-periodic sequences
case {3,4,5,6,8}
switch choice_p
  case {3,6,8}
    s_xx = ifft(fft(FS).*conj(fft(FS)));
  case 4
    mls(find(mls==0)) = -1;
    s_xx = ifft(fft(mls).*conj(fft(mls)));
  case 5
    s_xx = ifft(fft(TM).*conj(fft(TM)));
end

%Convert to an auto-correlation function (ACF)
Ns = length(s_xx);
Ns2 = floor(Ns/2)+1;
if Ns/2 == round(Ns/2)
  R_xx = [s_xx(Ns2:Ns),s_xx(1:Ns2)]./max(s_xx); %Convert ACVF to ACF and unwrap
  lags = linspace(-(Ns2-1),Ns2-1,Ns+1); %Delay array
else
  R_xx = [s_xx(Ns2+1:Ns),s_xx(1:Ns2)]./max(s_xx); %Convert ACVF to ACF and unwrap
  lags = linspace(-(Ns2-1),Ns2-1,Ns); %Delay array
end

%Plot ACF
figure
plot(lags,R_xx) %Plot ACF
xlabel('Delay, \tau') %x-axis label
ylabel('Auto-correlation function, R_{xx}(\tau)') %y-axis label
grid on %Turn gridlines on

%Apply a title
switch choice_p
  case {3,8}
    title(['The Auto-correlation of the Fibonacci sequence (order ',num2str(choice_f),')'])
  case 5
    title(['The Auto-correlation of the Thue-Morse sequence (order ',num2str(choice_tm),')'])
  case 6
    title(['The Auto-correlation of the symmetric Fibonacci sequence (order ',num2str(choice_f),')'])
end

case 7

%Calculate band structure for infinite perfectly periodic system
K = (1/s)*acos(xi); %Bloch wave-vector
%Select only the real values to plot
Kbs = K(find(imag(K)==0))/(pi/s); %Band structure K values normalised to pi/period
fbs = f(find(imag(K)==0)); %Corresponding frequency values

figure
plot(Kbs,fbs,'bo','MarkerSize',2,'MarkerFaceColor','b'); %Plot the wave vector vs omega for both positive and negative K
hold on
plot(-Kbs,fbs,'bo','MarkerSize',2,'MarkerFaceColor','b');
hold off
xlim([-1.1,1.1]) %x-axis limits
xlabel('Bloch wavevector, K (normalised to \pi/\Lambda)') %x-axis label
ylabel('Frequency, (Hz)') %y-axis label
grid on %Turn gridlines on

%Add a title to the graph
D.2 Functions used in main model

Periodic layers transfer function generation

%-------------------------------------------------------------------------
%periodic_layers_tf.m generates the transfer function coefficients of a
%system with n periodic layers
%-------------------------------------------------------------------------
%Function takes the arguments

% n: Number of periods
% k: Wavenumber
% a: Half length of section B
% s: Total period length (A+B)
% S1: Area of A sections
% S2: Area of B sections
% rho0: Density of medium
% c: Speed of sound in medium
% omega: Angular frequencies array used in the model
% choice_ml: Whether to include mass loading effects
%-------------------------------------------------------------------------
function [P_N11 P_N12 P_N21 P_N22 xi eta] = periodic_layers_tf(n,k,a,s,S1,S2,rho0,c,omega,choice_ml);

%Area coefficient definitions
eta = S1/S2; % Area ratio

% Define sequence of area variations
S = [S2 S1];

% Define length between perturbations
l = s-2*a;

% Define ABCD coefficient functions
A = @(k,Z,d) [cos(k*d)];
B = @(k,Z,d) [j*Z*sin(k*d)];
C = @(k,Z,d) [(j/Z)*sin(k*d)];
D = @(k,Z,d) [cos(k*d)];

% Define TM for first section of waveguide
P_11 = A(k,rho0*c/S2,2*a);
P_12 = B(k,rho0*c/S2,2*a);
P_21 = C(k,rho0*c/S2,2*a);
P_22 = D(k,rho0*c/S2,2*a);

% Include mass loading effects if required
switch choice_ml
    case 1
        % Define ABCD coefficients for mass loading
        ML_11 = ones(size(omega));
        ML_12 = j*omega*4;
        ML_21 = zeros(size(omega));
        ML_22 = ones(size(omega));

        % Concatenate mass loading matrix before and after perturbation
        [P_11 P_12 P_21 P_22] = concat([ML_11,ML_12,ML_21,ML_22,P_11,P_12,P_21,P_22]);
        [P_11 P_12 P_21 P_22] = concat(P_11,P_12,P_21,P_22,ML_11,ML_12,ML_21,ML_22);
end

% Concatenate TM for second half of period
[ML_11 ML_12 ML_21 ML_22] = [ML_11,ML_12,ML_21,ML_22,ML_11,ML_12,ML_21,ML_22];

% Bloch phase calculation (for infinite periodic system)
xi = 0.5*(P_11+P_22);  %Cosine of Bloch phase
gamma = acos(xi);  %Bloch phase

% Chebychev coefficient function
U = @(N,nu) sin((N+1)*acos(nu))./sin(acos(nu));

% Calculate values for multiple periods if n > 1
if n > 1
    % Total transfer function matrix elements
    P_N11 = (P_11.*U(n-1,xi))-U(n-2,xi);
    P_N12 = P_12.*U(n-1,xi);
    P_N21 = P_21.*U(n-1,xi);
    P_N22 = (P_22.*U(n-1,xi))-U(n-2,xi);

    % Correct values where xi = 1
    P_N11(find(xi==1)) = (P_11(find(xi==1))*n)-(n-1);
    P_N12(find(xi==1)) = P_12(find(xi==1))*n;
    P_N21(find(xi==1)) = P_21(find(xi==1))*n;
    P_N22(find(xi==1)) = (P_22(find(xi==1))*n)-(n-1);
else
    P_N11 = P_11;
    P_N12 = P_12;
    P_N21 = P_21;
    P_N22 = P_22;
end

Varying thickness layers transfer function generation

function [P_N11 P_N12 P_N21 P_N22 xi eta] = var_thick_layers_tf(n,S,k,a,s,S1,S2,rho0,c,omega,choice_ml);

% Area coefficient definitions
eta = S1/S2;  % Area ratio

% Calculate xi for a single period for comparison using fn periodic_layers_tf.
[P_N11 P_N12 P_N21 P_N22 xi] = periodic_layers_tf(1,k,a,s,S1,S2,rho0,c,omega,choice_ml);
clear P_N11 P_N12 P_N21 P_N22

A = @(k,Z,d) [cos(k*d)];
B = @(k,Z,d) [(j/Z)*sin(k*d)];
C = @(k,Z,d) [j*Z*sin(k*d)];
% Define ABCD coefficients for mass loading for use if required
ML_11 = ones(size(omega));
ML_12 = j*omega*4;
ML_21 = zeros(size(omega));
ML_22 = ones(size(omega));

%% Define TM for first section of waveguide
if S(1) < 0 % If sequence starts with one or more A layers
    P_N11 = A(k,rho0*c/S1,l*abs(S(1)));
P_N12 = B(k,rho0*c/S1,l*abs(S(1)));
P_N21 = C(k,rho0*c/S1,l*abs(S(1)));
P_N22 = D(k,rho0*c/S1,l*abs(S(1)));
else % If sequence starts with one or more B layers
    P_N11 = A(k,rho0*c/S2,2*a*abs(S(1)));
P_N12 = B(k,rho0*c/S2,2*a*abs(S(1)));
P_N21 = C(k,rho0*c/S2,2*a*abs(S(1)));
P_N22 = D(k,rho0*c/S2,2*a*abs(S(1)));
% Include mass loading effects if required
switch choice_ml
    case 1
        % Concatenate mass loading matrix before and after perturbation
        [P_N11 P_N12 P_N21 P_N22] = concat(ML_11,ML_12,ML_21,ML_22,P_N11,P_N12,P_N21,P_N22);
        [P_N11 P_N12 P_N21 P_N22] = concat(P_N11,P_N12,P_N21,P_N22,ML_11,ML_12,ML_21,ML_22);
    end
end

% Concatenate other transfer matrices in for each section
for c1 = 2:length(S)
    if S(c1) < 0 % If one or more A layers is concatenated
        [P_N11 P_N12 P_N21 P_N22] = concat(P_N11,P_N12,P_N21,P_N22,A(k,rho0*c/S1,l*abs(S(c1))));
        B(k,rho0*c/S1,l*abs(S(c1))),C(k,rho0*c/S1,l*abs(S(c1))),D(k,rho0*c/S1,l*abs(S(c1))));
    else % If one or more B layers is concatenated
        P_B11 = A(k,rho0*c/S2,2*a*abs(S(c1)));
P_B12 = B(k,rho0*c/S2,2*a*abs(S(c1)));
P_B21 = C(k,rho0*c/S2,2*a*abs(S(c1)));
P_B22 = D(k,rho0*c/S2,2*a*abs(S(c1)));
% Include mass loading effects if required
switch choice_ml
    case 1
        % Concatenate mass loading matrix before and after perturbation
        [P_B11 P_B12 P_B21 P_B22] = concat(ML_11,ML_12,ML_21,ML_22,P_B11,P_B12,P_B21,P_B22);
        [P_B11 P_B12 P_B21 P_B22] = concat(P_B11,P_B12,P_B21,P_B22,ML_11,ML_12,ML_21,ML_22);
    end
end
% If multiple periods of the sequence are being modelled, generate new
% transfer function elements to include this periodicity
if n > 1
% Chebyshev coefficient function
U = @(n,nu) sin((n+1)*acos(nu))./sin(acos(nu));
x = x + 0.5*(P_N11+P_N22);
P_N11 = P_N11.*U(n-1,x)-U(n-2,x);
P_N12 = P_N12.*U(n-1,x);
P_N21 = P_N21.*U(n-1,x);
P_N22 = P_N22.*U(n-1,x)-U(n-2,x);
end

% Concatenation of transfer matrices
%-------------------------------------------------------------------------
% concat.m concatenates two 2x2 matrices where each element of the matrix
% can be multi-valued (e.g. have values for multiple frequencies)
%-------------------------------------------------------------------------

% D MATLAB models
D = @(k,Z,d) [cos(k*d)];
%Philip D. King
%Research project, MSc Audio Acoustics
%University of Salford, 2006.
%--------------------------------------------
%Function takes the arguments
% % A11,A12,A21,A22: Elements of matrix A
% % B11,B12,B21,B22: Elements of matrix B
%--------------------------------------------
function [M11 M12 M21 M22] = concat(A11,A12,A21,A22,B11,B12,B21,B22);

%Concatenate elements together
M11 = (A11.*B11)+(A12.*B21);
M12 = (A11.*B12)+(A12.*B22);
M21 = (A21.*B11)+(A22.*B21);
M22 = (A21.*B12)+(A22.*B22);

Full width half maximum peak calculation
%-------------------------------------------------------------------------
%fwhm.m estimates the FWHM (-3dB point) of a peak in y centred at a
%specified frequency f_c
%-------------------------------------------------------------------------
%Philip D. King
%Research project, MSc Audio Acoustics
%University of Salford, 2006.
%------------------------------------------
%Function takes the arguments
% % y: Function of which to calculate fwhm
% % f: Frequency array
% % f_c: Frequency of peak of y
%------------------------------------------
function [delta_f,OK] = fwhm(y,f,f_c);

%Find index of array closest to f_c (maximum of peak)
f_fc = abs(f-f_c);
N_fc = find(f_fc == min(f_fc));

%Calculate amplitude at f_c
y_fc = y(N_fc);
%Calculate amplitude at -3dB point
y_fu = 0.5*y_fc;
%Find index of array closest to -3dB frequency
delta_y = y-y_fu;
c1 = N_fc;
while delta_y(c1) > 0
    c1 = c1+1;
end

%Calculate bandwidth (FWHM)
delta_f = 2*(f(c1)-f(N_fc));

%Specify if resolution sufficient to give accurate results
if c1-N_fc<5
    OK = 0;
else
    OK = 1;
end

Periodic fractal system
%-------------------------------------------------------------------------
%fractal.m generates the Transfer function coefficients of a fractal system
%with n periodic layers
%-------------------------------------------------------------------------
% %Philip D. King %Research project, MSc Audio Acoustics %University of Salford, 2006. % %--------------------------------------------------------------- %Function takes the arguments % % n: Number of periods % k: Wavenumber % a: Half length of section B % s: Total period length (A+B) % S1: Area of A sections % S2: Area of B sections % delta_S: Fractal change in area % fr: Order of fractal % rho0: Density of medium % c: Speed of sound in medium %---------------------------------------------------------------

function [P_N11 P_N12 P_N21 P_N22 xi eta] = fractal(n,k,a,s,S1,S2,delta_S,fr,rho0,c);

%Area coefficient definitions
eta = S1/S2; %Overall area ratio

%Determine changes in area due to fractal nature
dS = zeros(1,(2*fr)-1);
for c1 = 2:fr
dS(c1) = (c1-1)*delta_S;
end
dS((fr+1):length(dS)) = fliplr(dS(1:fr-1));

%Area array
S = [S2+dS S1+dS];

%Length of each section
seq(1,1) = 0;
for c1 = 2:fr
    seq(c1,:) = seq(c1-1,:);
    seq(c1,c1-1) = seq(c1,c1-1)+1;
    seq(c1,c1) = seq(c1,c1-1);
end
seq = [seq(fr,:) fliplr(seq(fr,1:fr-1))];

%Length array
l_frac = 3.^[seq];

%Length array
l_sect = [2*a./l_frac (s-(2*a))./l_frac];

%Impedance of each section
Z = rho0*c./S;

%Define ABCD coefficient functions
A = @(k,Z,l) [cos(k*l)];
B = @(k,Z,l) [(j/Z)*sin(k*l)];
C = @(k,Z,l) [j*Z*sin(k*l)];
D = @(k,Z,l) [cos(k*l)];

%Define single period ABCD transfer matrix elements %Define first layer
P_11 = A(k,Z(1),l_sect(1));
P_12 = B(k,Z(1),l_sect(1));
P_21 = C(k,Z(1),l_sect(1));
P_22 = D(k,Z(1),l_sect(1));

%Concatenate in subsequent layers
for c1 = 2:length(S)
    %[P_N11 P_N12 P_N21 P_N22] = concat(P_11,P_12,P_21,P_22,A(k,Z(1),l_sect(c1)),B(k,Z(c1),l_sect(c1)),%
    %C(k,Z(c1),l_sect(c1)),D(k,Z(c1),l_sect(c1)));
end

%Bloch phase calculation (for infinite periodic system)
xi = 0.5*(P_11+P_22); %Cosine of Bloch phase
gamma = acos(xi); %Bloch phase

%Chebychev coefficient function
U = @(N,nu) sin((N+1)*acos(nu))./sin(acos(nu));
% Calculate values for multiple periods if n > 1
if n > 1
  % Total transfer function matrix elements
  P_N11 = P_11.*U(n-1,xi)-U(n-2,xi);
  P_N12 = P_12.*U(n-1,xi);
  P_N21 = P_21.*U(n-1,xi);
  P_N22 = P_22.*U(n-1,xi)-U(n-2,xi);

  % Correct values where xi = 1
  P_N11(find(xi==1)) = (P_11(find(xi==1)).*n)-(n-1);
  P_N12(find(xi==1)) = P_12(find(xi==1)).*(n);
  P_N21(find(xi==1)) = P_21(find(xi==1)).*(n);
  P_N22(find(xi==1)) = (P_22(find(xi==1)).*(n))-(n-1);
else
  P_N11 = P_11;
  P_N12 = P_12;
  P_N21 = P_21;
  P_N22 = P_22;
end

Quasi-periodic fractal system

%_fractal2.m generates the Transfer function coefficients of a fractal system
% with n periodic layers

% Philip D. King
% Research project, MSc Audio Acoustics
% University of Salford, 2006.

function [P_N11 P_N12 P_N21 P_N22 xi eta] = fractal2(n,k,a,s,S1,S2,delta_S,FS,rho0,c);

% Area coefficient definitions
eta = S1/S2; % Overall area ratio

% Area variation
for c1 = 1:length(FS)
  if FS(c1) == 1
    S(c1) = S2;
  else
    S(c1) = S1;
  end
  if c1/2 == round(c1/2)
    S(c1) = S(c1)+delta_S;
  end
end

% Impedance of each section
Z = rho0*c./S;

% Define ABCD coefficient functions
A = @(k,Z,l) [cos(k*l)];
B = @(k,Z,l) [j*Z*sin(k*l)];
C = @(k,Z,l) [(j/Z)*sin(k*l)];
D = @(k,Z,l) [cos(k*l)];

% Define single period ABCD transfer matrix elements
% Define first layer
P_11 = A(k,Z(1),2*a);
P_12 = B(k,Z(1),2*a);
P_21 = C(k,Z(1),2*a);
P_22 = D(k,Z(1),2*a);

% Concatenate in subsequent layers
for c1 = 2:length(S)
  [P_11 P_12 P_21 P_22] = concat(P_11,P_12,P_21,P_22,A(k,Z(c1),2*a),B(k,Z(c1),2*a),C(k,Z(c1),2*a),D(k,Z(c1),2*a));
end

% Bloch phase calculation (for infinite periodic system)
xi = 0.5*(P_11+P_22); % Cosine of Bloch phase
gamma = acos(xi); % Bloch phase

% Chebychev coefficient function
U = @(N,nu) sin((N+1)*acos(nu))./sin(acos(nu));

% Calculate values for multiple periods if n > 1
if n > 1
    % Total transfer function matrix elements
    P_N11 = P_11.*U(n-1,xi)-U(n-2,xi);
    P_N12 = P_12.*U(n-1,xi);
    P_N21 = P_21.*U(n-1,xi);
    P_N22 = P_22.*U(n-1,xi)-U(n-2,xi);
    % Correct values where xi = 1
    P_N11(find(xi==1)) = (P_11(find(xi==1)).*n)-(n-1);
    P_N12(find(xi==1)) = P_12(find(xi==1)).*(n);
    P_N21(find(xi==1)) = P_21(find(xi==1)).*(n);
    P_N22(find(xi==1)) = (P_22(find(xi==1)).*(n))-(n-1);
else
    P_N11 = P_11;
    P_N12 = P_12;
    P_N21 = P_21;
    P_N22 = P_22;
end

Fibonacci structure sequence generation

%-------------------------------------------------------------------------
% Fib.m generates a Fibonacci structure sequence of order n
%-------------------------------------------------------------------------
% Philip D. King
% Research project, MSc Audio Acoustics
% University of Salford, 2006.
%-------------------------------------------------------------------------
% Function takes the arguments
% n: Order of sequence
%-------------------------------------------------------------------------
function [FS1 S] = Fib(n);

% Define the Fibonacci sequence
F = [1 1];
for c1 = 3:n+1
    F(c1) = F(c1-2)+F(c1-1);
end

% Define the sequence in terms of 1 and 0 for the A and B sections
FS1(1,1) = 1;
for c1 = 3:n+1
    FS1(c1,1) = F(c1-2)+F(c1-1);
end

% Define the sequence in terms of 1 and 0 for the A and B sections
FS1(1,1) = 1;
FS1(2,1) = 0;
for c1 = 3:n+1
    FS1(c1,1:F(c1)) = [FS1((c1-2),1:F(c1-2)) FS1((c1-1),1:F(c1-1))];
end

% Define actual sequence based on user choice of order
FS1 = FS1(n+1,:);

% Recast FS in terms of 1's and -1's
FS = FS1;
FS(find(FS==0)) = -1;

% Initialisate some counters
c1 = 1;
c2 = 1;

N = length(FS); % Length of sequence

% Group all 1's and -1's into higher integers representing how many sections
% are concatenated together in sequence
for c1 = 1:N
if FS(c1) < 0
    if c1 == 1 | FS(c1-1) > 0
        S(c2) = -1;
    else
        S(c2) = S(c2)-1;
    end
    if c1 == N | FS(c1+1) > 0
        c2 = c2+1;
    end
else
    if c1 == 1 | FS(c1-1) < 0
        S(c2) = 1;
    else
        S(c2) = S(c2)+1;
    end
    if c1 == N | FS(c1+1) < 0
        c2 = c2+1;
    end
end

%-------------------------------------------------------------------------
%Thue_Morse.m generates a Thue-Morse strucutre sequence of order n
%-------------------------------------------------------------------------
%
%Philip D. King
%Research project, MSc Audio Acoustics
%University of Salford, 2006.
%
%Function takes the arguments
%
% n: Order of sequence
%-------------------------------------------------------------------------

function [TM S] = Thue_Morse(n);

%Define the Thue-Morse sequence in terms of 1 and 0 for the A and B sections
TM(1,1) = 0;
for c1 = 2:n+1
    for c2 = 1:2^(c1-2)
        if TM(c1-1,c2) == 0
            TM(c1,(c2*2)-1:c2*2) = [0 1];
        elseif TM(c1-1,c2) == 1
            TM(c1,(c2*2)-1:c2*2) = [1 0];
        end
    end
end

%Define actual sequence based on user choice of order
TM = TM(n+1,:);

%Recast TM sequence in terms of 1's and -1's
TM1 = TM;
TM1(find(TM1==0)) = -1;

%Initialise some counters
c1 = 1;
c2 = 1;
N = length(TM1); %Length of sequence

%Group all 1's and -1's into higher integers representing how many sections
%are concatenated together in sequence
for c1 = 1:N
    if TM1(c1) < 0
        if c1 == 1 | TM1(c1-1) > 0
            S(c2) = -1;
        else
            S(c2) = S(c2)-1;
        end
        if c1 == N | TM1(c1+1) > 0
            c2 = c2+1;
        end
    end
end
c2 = c2+1;
else
    if c1 == 1 | TM1(c1-1) < 0
        S(c2) = 1;
    else
        S(c2) = S(c2)+1;
    end
    if c1 == N | TM1(c1+1) < 0
        c2 = c2+1;
    end
end

---

**MLS structure sequence generation**

```matlab
%-------------------------------------------------------------------------
%MLS_layer.m generates an MLS layer sequence of order n
%-------------------------------------------------------------------------

%Philip D. King
%Research project, MSc Audio Acoustics
%University of Salford, 2006.
%
%------------------------------------------
%Function takes the arguments
% % n: Order of sequence
%------------------------------------------
function [MLS,S,N] = MLS_layer(n);

%Taps for different order generation
taps = [0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1
        0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1
        0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 1
        0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1
        0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 0 1
        0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 0 1
        0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 1
        0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 1
        0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 1
        0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 1
        0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 0 0 1
        0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1
        1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1];
N = 2^n-1; %length of sequence (of order n)
%Define taps to use for sequence of order n
b = (taps(n-1,21-n:21))'; %Selects appropriate taps from table
%Set latches with initial values
x = zeros(n,1);
x(1)=1;
%Perform sum (modulo 2) from Eqn (5) of DSP study guide (T. J. Cox, 2006)
for m=n+1:N
    x(m) = mod(sum(flipud(x(m-n:m-1)).*b(2:n+1)),2);
end
%Take transpose to give row vector
MLS = x';
%Recast MLS in terms of 1's and -1's
x = MLS;
x(find(x==0)) = -1;
```
% Initialise some counters
    c1 = 1;
    c2 = 1;

    % Group all 1's and -1's into higher integers representing how many sections
    % are concatenated together in sequence
    for c1 = 1:N
        if x(c1) < 0
            if c1 == 1 | x(c1-1) > 0
                S(c2) = -1;
            else
                S(c2) = S(c2)-1;
            end
            if c1 == N | x(c1+1) > 0
                c2 = c2+1;
            end
        else
            if c1 == 1 | x(c1-1) < 0
                S(c2) = 1;
            else
                S(c2) = S(c2)+1;
            end
            if c1 == N | x(c1+1) < 0
                c2 = c2+1;
            end
        end
    end

Butterworth filter model
%-------------------------------------------------------------------------
% digital_filter.m models the transmission of a digital Butterworth filter
% of specified order.
%-------------------------------------------------------------------------
% Philip D. King
% Research project, MSc Audio Acoustics
% University of Salford, 2006.

    clear all
    close all
    clc

    % Define sampling frequency
    fs = 6000;

    % Define lower and upper frequency -3dB points
    f1 = 762; % Lower
    fu = 1231; % Upper
    Wn = [f1/(fs/2) fu/(fs/2)]; % In terms of Nyquist

    % Order of filter
    n = 32;

    % Generate filter coefficients
    [b,a] = butter(n/2,Wn,'stop');

    % Determine frequency response
    [H,w] = freqz(b,a);

    % Frequency array
    f = (w/pi)*fs/2;

    % Plot the filter (transmission coefficient)
    figure
    plot(f,abs(H).^2,'m');
    xlabel('Frequency (Hz)'); % x-axis label
    ylabel('Transmission coefficient, \alpha_t'); % y-axis label
    grid on % Turn gridlines on

D.3 Genetic algorithm model
%-------------------------------------------------------------------------
GA.m uses a genetic algorithm to search for the combination of waveguide layers to give a wide band gap at the design frequency

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Research project, MSc Audio Acoustics
University of Salford, 2006.

clear all
close all
clc

Initial specifications
Physical constants of medium (assuming T=20degC, p=1atm)
c = 343; %Speed of sound in air (ms^-1)
rho0 = 1.21; %Density of air (Kgm^-3)

Design parameters of the stack
f_c = 1300; %Design frequency of the structure
N = 20; %Define the number of layers

Areas of each section
S1 = (54e-3)^2;
S2 = (38e-3)^2;
S1 = 4;
S2 = 1;

Calculate remaining parameters
lambda_c = c/f_c; %Corresponding design wavelength
s = lambda_c/2; %Corresponding period length
a = s/4; %Half the length of the perturbation
l = s-(2*a); %Length of waveguide between perturbations

Frequency range of model
f = linspace(600,2000,1451);
omega = 2*pi*f; %Angular frequency range
k = (2*pi*f)/c; %Wavenumber in air

Specification of the genetic algorithms
npop = 25; %number of individuals in population
die = round(npop/6); %number to die per cycle
mutate_rate = 0.7; %Chance of mutation

Specify starting population
pop = (rand(npop,N)*2-1); %randomise population (between -1 and 1)

Display options to screen
disp('Starting population:');
disp('---------------------------------------');
disp('(1) Random starting population');
disp('(2) Include best solution from last run');
disp('---------------------------------------');
while choice_s < 1 | choice_s > 2 %Ensure that choice is made from list of options
choice_s = input('Choose starting population type: '); %Select starting population type
end

Specify starting population based on user choice
switch choice_s,
  case 2, %If option (2) selected
    load 'strt.mat'; %Load in best solution from last run
    pop(1,:) = strt;%Specify starting population to be best solution from last run
end

Calculate fitness of initial population
for j=1:npop
  %For each member of the population
  [P_N11(j,:), P_N12(j,:), P_N21(j,:), P_N22(j,:)] = var_thick_layers_tf(1,pop(j,:),k,a,s,S1,S2,rho0,c,omega,2);
end

Calculate transmission (intensity) coefficients
alpha_t = abs(1./(0.5*(P_N11+(P_N12/(rho0*c/S1))+((rho0*c/S1)*P_N21)+P_N22))).^2;

Calculate figure of merit - Mean of transmission coefficient in range
%specified (minimum corresponds to least transmission in this region)
fitness = mean(alpha_t,2);

%Define maximum transmission allowed in frequency region specified
tollerance = 0.01;

%Adjust fitness values to reflect tollerance
for j = 1:npop
    if max(alpha_t(j,:)) > tollerance
        %Increase fitness value if transmission greater than allowed
        fitness(j) = 100*fitness(j);
    end
end

%Store the minimum, maximum and mean values of the fitness
fitness_record(1,1) = min(fitness); %Minimum fitness
fitness_record(1,2) = max(fitness); %Maximum fitness
fitness_record(1,3) = mean(fitness); %Mean fitness
ncount = 1; %Initialise a counter

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Perform iterations until solutions converge (difference between minimum
%and maximum fitness becomes very small) AND fitness < 1 (within tollerance)
while (fitness_record(ncount,2)-fitness_record(ncount,1))>1e-12 | fitness_record(ncount,1) > 1
    %Form cumulative probability distribution to make decisions for breeding and dying
    fm = min(fitness); %Calculate the minimum fitness
    fcm_die = cumsum(fitness-fm); %Cumulatively sum the fitness values minus minimum value
    fcm_die = fcm_die/fmx; %Normalise values to give cumulative distribution for dying selection
    fm = max(fitness); %Calculate the maximum fitness
    fcm_breed = cumsum(fm-fitness); %Cumulatively sum the fitness values minus maximum value
    fcm_breed = fcm_breed/fmx; %Normalise values to give cumulative distribution for breeding selection

    %Randomly pick those who die, least fit most likely to die (from cumulative distribution)
    for m=1:ndie %Run as many times as number that die each cycle
        rd = rand(1,1)*0.99999; %Generate a random number (from 0 to very nearly 1)
        idie = 1; %Initialise index of person to die
        while rd>fcm_die(idie) %While random number is greater than the cumulative probability of person idie
            idie = idie + 1; %then increment index of person to die by one
        end
        %Breed a new off-spring to take place of dead one
        %Choose parents randomly, but fittest are most likely to breed (from cumulative distribution)
        for n=1:2 %Pick each parent seperately
            rd = rand(1,1)*0.9999; %Generate a random number (from 0 to very nearly 1)
            ibreed(n) = 1; %Initialise index of person to breed
            while rd>fcm_breed(ibreed(n)) %While random number is greater than the cumulative probability of person to breed
                ibreed(n) = ibreed(n) + 1; %then increment index of person to breed by one
            end
        end
        %Choose genes randomly from between parent values
        difference = pop(ibreed(2),:)-pop(ibreed(1),:); %Difference between parent 1 and 2 genes
        %Genes chosen randomly to be somewhere between parent genes
        pop(idie,:) = pop(ibreed(1),:)+(rand(1,N).*difference);
        %Chance of mutation
        %Determine number of genes to mutate based on mutation rate and a random number
        nmutate = floor(mutate_rate/rand(1,N));
        if nmutate > N %Ensure not more than total number of genes mutate
            nmutate = N;
        end
        nmut(ncount) = nmutate; %Store number of genes which mutate
    end
end

%Determine which genes mutate
clear imut
imut(1) = 0;
if nmutate == N %If all genes mutate
    imut(2:N+1) = [1:N]; %Specify a matrix of all genes to mutate
else
    for cl = 2:nmutate+1
        %Code for mutating genes
    end
end
% MATLAB models
imut(c1) = round(N*rand(1,1)); %Generate randomly which gene mutates
while any(imut(1:c1-1) == imut(c1)) %If the gene has already been selected for mutation
    imut(c1) = mod(imut(c1)+1,N); %Increment the gene number to mutate
end
end
for mm=1:nmutate %For each gene to mutate
    j = imut(mm+1);
    %Calculate magnitude of difference of parent 1 and 2 gene
    diff = abs(pop(ibreed(1),j)-pop(ibreed(2),j));
    %Generate a plus or minus 1 (i.e. determine wheter to add to or
    %subtract from previous gene value during mutation)
    pm = (2*round(rand(1,1)))-1;
    %Add or subtract a random number generated from Gaussian
    %distribution centred on |parent 1-parent 2| to mutate the gene
    pop(idie,j) = pop(idie,j)+pm*((0.05)*randn(1,1)+diff);
end
%Generate transfer function matrix elements from function
%quasiperiodic_layers_tf.m for the new member of population
[P_N11(idie,:) P_N12(idie,:) P_N21(idie,:) P_N22(idie,:)] = var_thick_layers_tf(1,pop(idie,:),
    k,a,s1,s2,rho0,c,omega,2);
%Calculate transmission (intensity) coefficients
alpha_t(idie,:) = abs(1./(0.5*(P_N11(idie,:)+(P_N12(idie,:)/(rho0*c/S1))+((rho0*c/S1)*P_N21(idie,:)))+P_N22(idie,:))).^2;
%Calculate figure of merit - Mean of transmission coefficient in range
%specified (minimum corresponds to least transmission in this region)
fitness(idie) = mean(alpha_t(idie,:));
%Adjust fitness values to reflect tollerance
if max(alpha_t(idie,:)) > tollerance
    fitness(idie) = 2*fitness(idie);
end
%Increment the counter
ncount = ncount + 1;
%Store the new minimum, maximum and mean values of the fitness
fitness_record(ncount,1) = min(fitness); %Minimum value
fitness_record(ncount,2) = max(fitness); %Maximum value
fitness_record(ncount,3) = mean(fitness); %Mean value
if ncount == 2000 %Change mutate rate after a number of iterations
    mutate_rate = 0.1; %Chance of mutation
end
if ncount == 3000 %Only run until number of iterations reaches 5000
    break
end
%Select the population member to use as solution
ibest = find(fitness == min(fitness)); %Find the minimum fitness value (smallest mean sq error)
final_pop = pop(ibest(1),:); %Select best individual to use for final solution

%Plot of relevant results
figure
num = [1:ncount]; %Define an array of iteration number
semilogy(num,fitness_record(:,1),'r',num,fitness_record(:,2),'m',num,fitness_record(:,3),'g')
legend('Min','Max','Mean','0') %Apply a legend
xlabel('Iteration number') %x-axis label
ylabel('Fitness') %y-axis label
%Apply a title to the graph
title('Fitness variation with iterations')

figure
%Plot the transmission coefficient over the modelled frequency range
plot(f,alpha_t(ibest(1,:)))
xlabel('Frequency (Hz)') %x-axis label
ylabel('Transmission coefficient, $\alpha_t$') %y-axis label
grid on %Apply gridlines

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Save best solution so it can be loaded in for re-run of algorithm
strt = final_pop; %Re-define variable name for best solution
save 'strt.mat' strt %Save variable
Bibliography


