A Tutorial on SISO and MIMO Channel Capacities

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Abstract—This is a tutorial article presenting comparison between single input single output and multi input multi output wireless channel. We have compared Shannon ergodic capacity of SISO link and its different realizations with MIMO wireless channel. It is shown that wireless channel capacity increases linearly with the increase of multiple antennas both at transmitter and receiver. MIMO wireless link capacities when transmitter and receiver both have perfect knowledge of channel matrix, only transmitter has the channel matrix knowledge and when only receiver has the channel knowledge are presented. MIMO channel decomposition using eigen values and singular values decomposition methods is presented.

Key Words—Single Input Single Output (SISO), Multi Input Multi Output (MIMO), Signal to Noise Ratio (SNR), Shannon-capacity.

I. INTRODUCTION

In wireless system design, the most important part to understand and design is the wireless channel. It is time-varying random filter which effects the capacity and successful transmission of information. In this article we have studied the characteristics of SISO and MIMO wireless channels. In ideal conditions, Shannon capacity of SISO channel depends upon available bandwidth (B), transmit power (P) and interference from noise (N). In order to achieve maximum SISO capacity for SISO channel, we have to increase the B or P or reduce the noise level (N). In practical systems we have limitations on B and P i.e. fixed available spectrum and power constraint. Noise factor in wireless communication depends upon many factors including fading, shadowing, mobility of user and environment both. Hence we have a limited wireless capacity. In [1], authors have shown that higher spectral capacity and hence efficiency can be achieved by using multiple antennas at transmitter and receive. It was shown that capacity increases linearly with the increase of number of antennas. Using MIMO system, parallel transmit streams of single user or multiple users can be sent and received. Hence using these parallel sub-channels very high capacity can be achieved. It is then MIMO systems which will make giga-bit wireless systems a reality [2].

There is enormous research done for wireless channel capacity calculations and design, found in many wireless channel literature references. The contribution of this article is for our better understanding of wireless channel capacity and wireless channel characteristics and by no means to present any novel technique in this area of research. However the results of this work will be used in our future work on design of scheduling algorithms for single and multi hop wireless networks. Most of the capacity calculations and discussions in this work are based on the references [3], [4], [5], [6], and [7]. This reference list is not a comprehensive and exhaustive coverage of research in this field (later references will be provided in the article) but gives an authoritative knowledge on the subject.

The article is organized as follows. In section II, we describe the SISO system capacity and compare the results of Shannon capacity theorem with some other channel distributions. Section III, discusses the MIMO log-determinant capacity formula with assumptions that transmitter and receiver both have channel information, only transmitter has channel knowledge and then only receiver has channel knowledge. In section IV, we conclude our discussion.

II. SISO CHANNEL CAPACITY

Lets consider a typical communication system which consists of a transmitter, channel and receiver. The mathematical model of this communication system can be shown by using the notations in [1]

$$r = g * s + v$$  \hspace{1cm} (1)

where \( r \) is the received signal. At each point in time, this is an \( n_r \)-dimensional(no. of receive antennas) signal. \( g_t \) is the matrix channel impulse response. This matrix has \( n_t \) columns and \( n_r \) rows. The notation \( h_t \) is used for the normalized form of \( g_t \). \( s_t \) is the \( n_t \)-dimensional transmitted signal. \( v_t \) is the complex \( n_r \)-dimensional AWGN. For the simplicity we have omitted the subscript \( t \) and in this case of SISO channel we have \( n_t = n_r = 1 \). The ” * ” here means convolution. This is time discrete system model where noise is drawn i.i.d. from a Gaussian distribution of zero mean and variance \( N \). If the noise variance is zero then the receiver receives the transmitted symbol perfectly. If the noise variance is non-zero and there is no constraint on the input, we can choose and infinite subset of inputs arbitrarily far apart, so that they are distinguishable at the output with arbitrarily small probability of error. Such a scheme has an infinite capacity as well. Thus if the noise variance is zero or the input is unconstrained, the capacity of the channel is infinite. To derive the SISO channel capacity we consider information theoretic point of view. The information
capacity of the Gaussian channel with power constraint $P$ is

$$C = \max_{P(x):E[X^2] \leq P} I(X;Y)$$

(2)

We can calculate the information capacity by expanding $I(X;Y)$. Interested reader is referred to [3]. We can obtain the capacity formula for information capacity of Gaussian channel as:

$$C = \max_{P(x):E[X^2] \leq P} I(X;Y) = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right)$$

(3)

It is shown in [8] that the mean capacity of SISO system ($n_r = n_t = 1$) with a random complex channel gain $h_{11}$ is given as

$$C = E_H[\log_2(1 + \rho|h_{11}|^2)]$$

(4)

where $\rho$ is the average signal to noise ratio (SNR) at the receiver. If $|h_{11}|^2$ is Rayleigh, $|h_{11}|^2$ follows a chi-squared distribution with two degrees of freedom. Hence above equation can be written as:

$$C = E_H[\log_2(1 + \rho \chi^2_2)]$$

(5)

where $\chi^2_2$ is a chi-square distributed random variable with two degrees of freedom. In Fig. 1 we have shown the simulation results of mean SISO channel capacity for different distributions. Shannon capacity for Gaussian channel (solid line) is compared with other distributions. Channel is assumed with perfect estimation at both transmitter and receiver.

### III. MIMO Channel Capacity

Let’s consider a MIMO system with $n_r$ receive and $n_t$ transmit antennas as shown in Fig. 2. It is now well accepted fact that we can increase capacity without increasing the bandwidth and transmit power rather by just putting more antennas at transmitter and receiver side. The most important part to understand and deal the MIMO capacity is the channel matrix. Considering the impulse response between the $j$th ($j = 1, 2, ..., n_t$) transmit antenna and the $i$th ($i = 1, 2, ..., n_r$) receive antenna by $h_{i,j}(\tau, t)$, the MIMO channel is given by the $n_r \times n_t$ matrix $H(\tau, t)$ with

$$H(\tau, t) = \begin{bmatrix}
    h_{1,1}(\tau, t) & h_{1,2}(\tau, t) & \cdots & h_{1,n_r}(\tau, t) \\
    h_{2,1}(\tau, t) & h_{2,2}(\tau, t) & \cdots & h_{2,n_r}(\tau, t) \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{n_r,1}(\tau, t) & h_{n_r,2}(\tau, t) & \cdots & h_{n_r,n_r}(\tau, t)
\end{bmatrix}$$

(6)

The vector $[h_{1,1}(\tau, t) \ h_{2,1}(\tau, t) \ \cdots \ h_{n_r,1}(\tau, t)]^T$ is the spatio-temporal signature or channel induced by the $j$th transmit antenna across the receive antenna array. The random channel we have used is the Rayleigh model. In frequency domain, the channel is approximated by a complex matrix having the independent, identically distributed (iid) entries with zero mean and unit variance. A generalized capacity formula and a capacity lower-bound formula are mentioned below. This is for any $(n_r, n_t)$ MIMO system.

$$C = \log_2 \left( \det \left[ I_{n_r} + \left( \frac{\rho}{n_t} \right) HH^\dagger \right] \right) b/s/Hz$$

(7)

In this equation, "det" means determinant, $I_{n_r}$ means $n_r \times n_r$ identity matrix and "$^\dagger$" means transpose conjugate. The capacity lower bound for the $(n, n)$ case in terms of the independent chi-squared variable with two-degrees of freedom is as follows.

$$C \geq \sum_{k=1}^{n} \log_2 \left[ 1 + \left( \frac{\rho}{n} \right) \chi^2_{2k} \right] b/s/Hz$$

(8)

The capacity formula for optimum ratio combining or receive diversity $(n_r = n_t = n)$ is given as [1].

$$C = \log_2 \left[ 1 + \rho \chi^2_{2n} \right] b/s/Hz$$

(9)

It is worth noting from the results shown in Fig. 3 and Fig. 4 that capacity of wireless network increases as we increase the number of antennas.

### A. Statistical Properties of $H$

This section describes the some very important properties of channel matrix $H$ [5]. We describe in this section some statistics of $H$ like singular value decomposition. By diagonalizing the product matrix $HH^\dagger$ using eigen value decomposition, the matrix product is written as [8]

$$HH^\dagger = \Lambda E \Lambda^\dagger$$

(10)

where $E$ is the eigenvector matrix with orthonormal columns and $\Lambda$ is a diagonal matrix with the eigenvalues on the main
diagonal. Using this notation, the capacity of MIMO channels can be written as:

\[ C = E_H \left\{ \log_2[\det(I_{n_r} + \frac{\rho}{n_t} E \Lambda E^\dagger)] \right\} \]  \tag{11}

The matrix product \( HH^\dagger \) can also be described by using singular value decomposition (svd) on the channel matrix \( H \), written as:

\[ H = U \Sigma V^\dagger \]  \tag{12}

where \( U \) and \( V \) are unitary matrices of left and right singular vectors respectively, and \( \Sigma \) is a diagonal matrix with singular values on the main diagonal. All elements on the diagonal are zero except for the first \( k \) elements. The number of non-zero singular values \( k \) equals the rank of the channel matrix. Using SVD decomposition the MIMO channel capacity can be written as:

\[ C = E_H \left\{ \log_2[\det(I_{n_r} + \frac{\rho}{n_t} U \Sigma V^\dagger U^\dagger)] \right\} \]  \tag{13}

After diagonalizing the product matrix \( HH^\dagger \), the capacity formula of the MIMO channel now includes the unitary and diagonal matrices only. It is then easier to see that the total capacity of a MIMO channel is made up by the sum of parallel AWGN SISO sub-channels.

The number of parallel sub-channels is determined by the rank of the channel matrix. In general, the rank of the channel matrix is given by

\[ \text{rank}(H) = k \leq \min\{n_r, n_t\} \]  \tag{14}

Using the equation (14), together with the fact that the determinant of a unitary matrix is 1, the capacity expression can be written as:

\[ C = E_H \left\{ \sum_{i=1}^{k} \log_2(1 + \frac{\rho}{n_t} \lambda_i) \right\} \]  \tag{15}

\[ = E_H \left\{ \sum_{i=1}^{k} \log_2(1 + \frac{\rho}{n_t} \sigma_i) \right\} \]  \tag{16}

Where \( \lambda_i \) are the eigen values of the diagonal matrix \( \Lambda \) and \( \sigma_i \) are the squared singular values of the diagonal matrix \( \Sigma \). When the channel is known at the transmitter, the maximum capacity of a MIMO channel can be achieved by using the water-filling algorithm [3] on the transmit covariance matrix. The capacity is then given by

\[ C = E_H \left\{ \sum_{i=1}^{k} \log_2(1 + \epsilon_i \frac{\rho}{n_t} \lambda_i) \right\} \]  \tag{17}

\[ = E_H \left\{ \sum_{i=1}^{k} \log_2(1 + \epsilon_i \frac{\rho}{n_t} \sigma_i) \right\} \]  \tag{18}

Where \( \epsilon_i \) is a scalar, representing the portion of the available transmit power going into the \( ith \) sub-channel. Hence, by using the water-filling algorithm we can meet the total power constraint.

IV. CONCLUSION

This article describes the capacity calculations of SISO and MIMO systems under different channel assumptions. We have presented the Shannon capacity formula and compare the results with MIMO channel capacity formulas. It is shown by simulation results in this article that how channel capacity increases linearly by using MIMO system. We have discussed statistical analysis of the MIMO channel matrix. Singular value decomposition and eigen value decompositions are discussed. Channel capacity formulas in terms of eigenvectors and singular vectors are explained. The concept of total MIMO capacity as a sum of many parallel SISO channels is discussed. The future work is in detail study of water-filling algorithm and zero-forcing mechanism in the MIMO systems.

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REFERENCES


